M336 2000 Exam Solutions

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Part I (55 marks - 90 minutes)

Question 1.

(a) Tile types: [3, 6, 3, 6] [3, 3, 3, 3, 6] [3, 3, 3, 3, 3, 3]
Vertex types: (5, 6, 6) (4, 5, 5) (5, 5, 6) (4, 5, 5, 4, 5, 5)

(b) $f = t[\mathbf{p}]\lambda[A] \Rightarrow f^2 = t[\mathbf{p} + A\mathbf{p}]\lambda[A^2]$

$$A^2 = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \text{ and } \mathbf{p} + A\mathbf{p} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

so $f^2 = t[(-3,-1)]\lambda\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$

(a) HB 9-10
Don't bother to memorise which brackets to use.

(b) HB p5 Rule 4 with $\mathbf{p} = \mathbf{q}$ and $A = B$

Question 2.

(a) $(r^3s)^2 = r^3sr^3s = r^3r^{-3}ss = s^2 = r^4$

(b) $(r^3s)^3 = (r^3s)^2(r^3s) = r^4r^3s = r^7s \neq e$

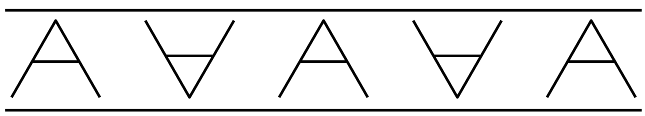
$(r^3s)^4 = [(r^3s)^2]^2 = (r^4)^2 = r^8 = e$ so r^3s has order 4.

(a) $sr = r^7s = r^{-1}s$, so $sr^3 = r^{-1}sr^2 = r^{-2}sr = r^{-3}s$

Question 3.

(a) v? yes; h? yes; so Type 6

(b)



v? yes; h? no; g? yes; so Type 7

HB 14 Algorithm

Question 4.

For $x, y\ (\neq e) \in G$, we have $x = x^{-1}$ and $y = y^{-1}$ (both order 2).
Then $xy = x^{-1}y^{-1} = (yx)^{-1} = yx$ (as yx has order 2). Hence G is Abelian.

Question 5.

(a) $G \cong D_3$, so $|G| = 6$.

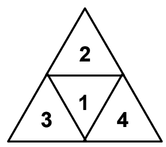
g	cycle type	cs(g)	number
e	(1)(2)(3)(4)	x_1^4	1
r, r^2	(1)(234)	x_1x_3	2
s, rs, r^2s	(1)(3)(24)	$x_1^2x_2$	3

Cycle index: $\frac{1}{6}\left(x_1^4 + 2x_1x_3 + 3x_1^2x_2\right)$

(b) Total with 3 colours: $\frac{1}{6}\left(3^4 + 2\cdot 3\cdot 3 + 3\cdot 3^2\cdot 3\right) = 30$

There are 3 colourings using only one colour, therefore 27 colourings with at least 2 colours.

(a) Label the diagram:



HB p27 Cycle Index Th'm.

NB: 4 triangles, so $x_a^bx_c^d$ must have $a\times b + c\times d = 4$.

Coefficients of the elements in the index should add up to $|G| = 6$.

Question 6.	
(a) $306 = 3\times85 + 51$ $85 = 1\times51 + 34$ $51 = 1\times34 + 17$ $34 = 2\times17 + 0$ so hcf(306, 85) = 17 (b) $17 = 51 - 34$ $= 51 - (85 - 51)$ $= 2\times51 - 85$ $= 2(306 - 3\times85) - 85$ $= 2\times306 - 7\times85$	Euclidean algorithm HB p22 (a) Check: $306 = 2 \times 3^2 \times 17$ $85 = 5 \times 17$ (b) Easy check: use your calculator on the final sum to make sure it does actually equal 17. (It's surprising how many people don't bother.)
Question 7.	
(a) $n_t(\Im) = 4$ (b) $\Im$ is periodic and tile–uniform with tile type = [3, 3, 4, 3, 4] So, by theorem 3.2, $n_v(\Im) = 4(^1/_3 + {}^1/_3 + {}^1/_4 + {}^1/_3 + {}^1/_4) = 6$ (c) There are 5 lines from each of 4 tile orbits, so a total of 20 lines in the tile–edge diagram. Two lines enter each edge orbit, so there must be 20/2 = 10 edge orbits.	(b) HB p29 Vertex orbit theorem
Question 8.	
(a) (b) $\begin{bmatrix}1&7&4\\2&1&0\\0&8&4\end{bmatrix} \to \begin{bmatrix}1&7&4\\0&-13&-8\\0&8&4\end{bmatrix}$ $R_2 \to R_2 - 2R_1$ $\to \begin{bmatrix}1&0&0\\0&-13&-8\\0&8&4\end{bmatrix}$ $C_2 \to C_2 - 7C_1$, $C_3 \to C_3 - 4C_1$ $\to \begin{bmatrix}1&0&0\\0&8&4\\0&-13&-8\end{bmatrix}$ $R_2 \leftrightarrow R_3$ $\to \begin{bmatrix}1&0&0\\0&4&8\\0&-8&-13\end{bmatrix}$ $C_2 \leftrightarrow C_3$ $\to \begin{bmatrix}1&0&0\\0&4&0\\0&-8&3\end{bmatrix}$ $C_3 \to C_3 - 2C_2$ $\to \begin{bmatrix}1&0&0\\0&4&0\\0&0&3\end{bmatrix}$ $R_3 \to R_3 + 2R_2$ (c) $A \cong Z_1 \times Z_4 \times Z_3 \cong Z_4 \times Z_3 \cong Z_{12}$ (Z_1 is trivial, 4, 3 are coprime)	(b) **Watchpoint:** Show your working. It's easy to make arithmetic errors, so you need to show what you were trying to do, even if you get it wrong.

Question 9.

(a) $\boldsymbol{c} = 2\boldsymbol{a} + \boldsymbol{b}$

The transition matrix from $\{\boldsymbol{a}, \boldsymbol{c}\}$ to $\{\boldsymbol{a}, \boldsymbol{b}\}$ is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ with integer entries and determinant equal to 1. Therefore $L(\boldsymbol{a}, \boldsymbol{c}) = L(\boldsymbol{a}, \boldsymbol{b})$.

(b) The transition matrix from $\{\boldsymbol{b}, \boldsymbol{c}\}$ to $\{\boldsymbol{a}, \boldsymbol{b}\}$ is $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

Determinant $= -2 \neq \pm 1$. Therefore $L(\boldsymbol{b}, \boldsymbol{c}) \neq L(\boldsymbol{a}, \boldsymbol{b})$.

(c) (i) $|\boldsymbol{a}| = \sqrt{5}$, $|\boldsymbol{b}| = 2\sqrt{10}$, $|\boldsymbol{c}| = 2\sqrt{5}$

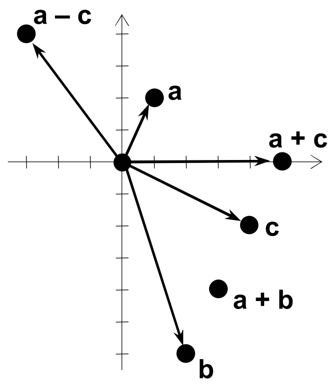
$L(\boldsymbol{a}, \boldsymbol{c}) = L(\boldsymbol{a}, \boldsymbol{b})$, and $\{\boldsymbol{a}, \boldsymbol{c}\}$ is a reduced basis for L.
$\boldsymbol{a} \cdot \boldsymbol{c} = 0$, so L is a rectangular lattice.

(ii) $\boldsymbol{a} + \boldsymbol{c} = (5, 0)$, $\boldsymbol{a} - \boldsymbol{c} = (-3, 4)$ and $|\boldsymbol{a} + \boldsymbol{c}| = |\boldsymbol{a} - \boldsymbol{c}| = 5$.

$(\boldsymbol{a} + \boldsymbol{c}) \cdot (\boldsymbol{a} - \boldsymbol{c}) = -15$, so angle $\neq \pi/2$, $\pi/3$ or $2\pi/3$.

Therefore $L(\boldsymbol{a} + \boldsymbol{c}, \boldsymbol{a} - \boldsymbol{c})$ is rhombic.

Picture, with lots of vertices.



Identity of lattices:
Th'm 1.2 HB p38

Classification of lattices:
Th'm 5.3, HB p40.

Question 10.

$|G| = pq$, and $Z(G)$ is a normal subgroup of G,
so $|Z(G)|$ divides pq, and $|Z(G)| \in \{1, p, q, pq\}$.

$|Z(G)| \neq pq$, since G is non–Abelian.

If $|Z(G)| = p$ then $|G/Z(G)| = q$ *and vice versa.* In either case the quotient group would be cyclic (prime order), and G would be Abelian – a contradiction.

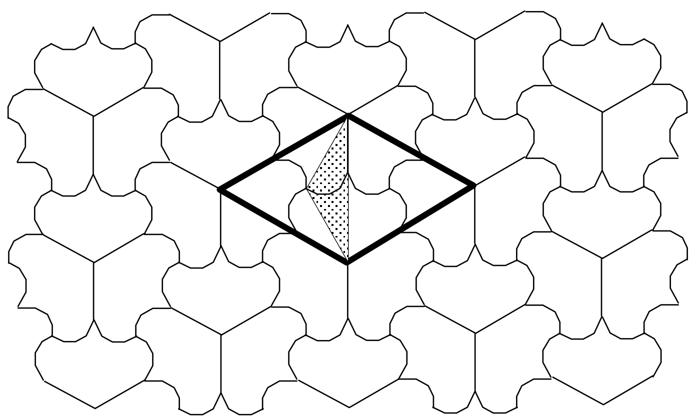
Therefore $|Z(G)| = 1$, and $Z(G)$ is the trivial subgroup.

Th'm 4.3, HB p37.

Question 11.

(a) Highest order of rotation? 3
Any reflections? Yes
3–centres all on axes? No
Therefore type $p31m$.

(b)



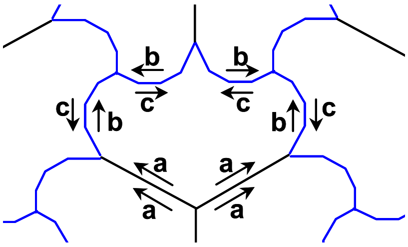
(b) The point group of type $p31m$ is D_3 (HB p46).

Since $|D_3| = 6$, the area of the generating region is $^1/_6 \times$ the area of the basic parallelogram. [Unit GE4, p8 final paragraph.]

Part II A (Geometry)

Question 12

(a)



(b) $a^{\to}\ b^{\to}\ c^{\to}\ c^{\leftarrow}\ b^{\leftarrow}\ a^{\leftarrow}$
$a^{\to}\ c^{\leftarrow}\ b^{\leftarrow}\ b^{\to}\ c^{\to}\ a^{\leftarrow}$

(c) $n_t(\Im) = 3$

(d) (i) 1
(ii) 3
(iii) 2
} $n_v(\Im) = 6$

By Euler's equation, $n_e(\Im) = n_t(\Im) + n_v(\Im) = 9$

(e) (i) There's a rotational symmetry about the centre of the tile (since the arrows in the tile symbol are all in the same direction). For the first 3 edges the arrows in the adjacency symbol are in the same direction as those in the tile symbol, and would remain in the same direction when the tile is rotated. The symbol should therefore be
$a^{\to}\ b^{\to}\ c^{\to}\ a^{\to}\ b^{\to}\ c^{\to}$
$b^{\to}\ c^{\to}\ a^{\to}\ b^{\to}\ c^{\to}\ a^{\to}$

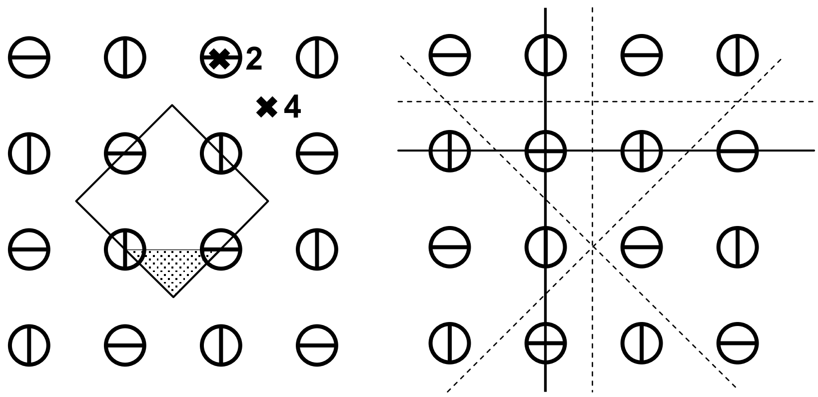
(ii) The 1st edge rotates to the 3rd edge (same label, same direction), so under the rotation, the 2nd edge rotates to the 4th edge. The tile symbol should therefore be $a^{\to}\ b^{\to}\ a^{\to}\ b^{\to}$.

(a) Incidence symbol
HB p31

(d) Euler's equation
Th'm 22, HB p29

Question 13.

(a)
(b)
(c)



(d) highest order? 4
reflections? yes
4 directions? no therefore *p4gm*

(e) The axes in the diagonals are all glides, ie they don't alternate with reflection axes, so they are rectangular by Theorem 2.7.

Check: From part (d) you know that the point group of *p4gm* is D_4 with order 8.
So the area of the generating region is $^1/_8 \times$ the area of the basic parallelogram.

(d) Algorithm HB p 45–46

(e) HB p44, Th'm 2.7.

Question 14.	
(a) $L(\boldsymbol{a}, \boldsymbol{b})$ is hexagonal, since $\lvert \boldsymbol{a} \rvert = \lvert \boldsymbol{b} \rvert = 2$, and $\boldsymbol{a} \cdot \boldsymbol{b} = 2 = \tfrac{1}{2}\lvert \boldsymbol{a} \rvert^2$. $\boldsymbol{c} = (0, 0, 1) + {}^1/_3\, \boldsymbol{a} + {}^1/_3\, \boldsymbol{b}$, where (0, 0, 1) is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$, so offset = $({}^1/_3, {}^1/_3)$. Vertical separation = 1 = $\tfrac{1}{2}\lvert \boldsymbol{a} \rvert$, so the lattice is **trigonal.** (b) (i) identity (A)(B)(C)(D) 8 order 3 rotations, (ABC) 3 order 2 rotations, (AD)(BC) (ii) G ≅ A_4, with order 12, and there are 12 short rods (labelled $a, b, \ldots, l$) and 6 long rods (labelled 1, 2, …, 6). **g** / **cycle type** / **cs(g)** / **number** e / $(a) \ldots (l)(1) \ldots (6)$ / x_1^{18} / 1 $r[2\pi/3]$ / $(alg)(bhc)(dek)(fij)(123)(456)$ / x_3^6 / 8 $r[\pi]$ / $(ai)(bj)(ck)(dj)(ef)(gh)(15)(26)(3)(4)$ / $x_1^2 x_2^8$ / 3 Cycle index: ${}^1/_{12}(x_1^{18} + 8\,x_3^6 + 3\,x_1^2 x_2^8)$ With m colours: ${}^1/_{12}(m^{18} + 8m^6 + 3m^{10})$ (iii) The number will be greater than $S(m)L(m)$, because a pair of colourings which are in the same conjugacy class in $S(m)$ may be matched with two different classes in $L(m)$ and vice versa.	(a) HB p56 Result 3.2 (b) (ii) NB: there are18 rods, so for cs(g) = $x_a^b x_c^d$ we must have $a \times b + c \times d = 18$. Coefficients of the elements in the index should add up to $\lvert G \rvert = 12$, and the divisor in the cycle index is ${}^1/_{12}$.

Part II B (Groups)

Question 15	
(a) <u>*Closure*</u>: Let $x = h_1 n_1$, $y = h_2 n_2 \in HN$, where $h_i \in H$, $n_i \in N$. We have $n_j h_i = h_i n'$ for some $n' \in N$ (N normal in G, so left cosets = right cosets). Then $xy = (h_1 n_1)(h_2 n_2) = h_1(n_1 h_2) n_2 = h_1(h_2 n') n_2 = (h_1 h_2)(n' n_2) \in HN$ since H and N are closed. <u>*Inverse*</u>: $(hn)^{-1} = n^{-1} h^{-1}$ in G, and $n^{-1} h^{-1} = h^{-1} n'' \in HN$, by the normality of N and the inverse properties of H and N. <u>*Identity*</u>: e is the identity of G. $e \in H$ and $e \in N$, (subgroups), so $ee = e \in HN$. Hence HN is a subgroup of G. (b) HN certainly contains H and N. Let K be *any* subgroup of G containing H and N. For any $hn \in HN$, $h \in H$ so $h \in K$, and $n \in N$ so $n \in K$, since K contains H and N. K is closed, so $hn \in K$. Hence $HN \subseteq K$. This is true for any subgroup K containing H and N, so HN is the smallest such subgroup. (c) If H and N are both normal, then for any $g \in G$ and any $hn \in HN$, $g(hn)g^{-1} = ghng^{-1} = gh(g^{-1}g)ng^{-1} = (ghg^{-1})(gng^{-1}) \in HN$ since $ghg^{-1} \in H$ (normal) and $gng^{-1} \in N$ (normal). Hence HN is normal in G.	(a) $n_j h_i \in Nh_i$ and $h_i n'$ is the equivalent element in $h_i N$ since $Nh_i = h_i N$. (b) Logic notation: $[(hn \in HN) \Rightarrow (hn \in K)]$ $\Rightarrow (HN \subseteq K)$ (c) Note the old trick of lobbing in $g^{-1}g$ to separate the elements into conjugates. *Alternative proof*: $g(HN) = (gH)N = (Hg)N = H(gN) = H(Ng) = (HN)g$ since H and N are normal.

Question 16.

(a) $|G| = 80 = 2^4 \times 5$

prime	factors	label
5	5	5a
2	2^4	2a
	2×2^3	2b
	$2^2 \times 2^2$	2c
	$2 \times 2 \times 2^2$	2d
	$2 \times 2 \times 2 \times 2$	2e

There are $1 \times 5 = 5$ possible groups.

label	*p*-primary form	canonical form
2a5a:	$Z_{16} \times Z_5$	Z_{80}
2b5a:	$Z_2 \times Z_8 \times Z_5$	$Z_2 \times Z_{40}$
2c5a:	$Z_4 \times Z_4 \times Z_5$	$Z_4 \times Z_{20}$
2d5a:	$Z_2 \times Z_2 \times Z_4 \times Z_5$	$Z_2 \times Z_2 \times Z_{20}$
2e5a:	$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_5$	$Z_2 \times Z_2 \times Z_2 \times Z_{10}$

(b) If the group contains an element of order 8 then it contains a cyclic subgroup of order 8 (isomorphic to Z_8), and its 2–primary component must contain such a subgroup. The only groups satisfying this condition are $Z_{16} \times Z_5 \cong Z_{80}$ and $Z_2 \times Z_8 \times Z_5 \cong Z_2 \times Z_{40}$.

(c) $Z_{20} \cong Z_4 \times Z_5$. The only group which does not possess such a subgroup is $Z_2 \times Z_2 \times Z_2 \times Z_{10}$, whose 2–primary components consists of groups of order 2. All the other groups have cyclic subgroups of orders 4 and 5, as can be seen from the p–primary form, and therefore possess a subgroup isomorphic to Z_{20}.

(a) Canonical decomposition HB p33 Theorem 3.2
p–primary decomposition HB p38

Question 17.

(a) $|G| = 2^3 \times p, p > 3$. Let n_p be the number of Sylow *p*-subgroups of *G*.

i) $p = 5$: $n_5 \equiv 1 \pmod 5$ and $n_5 \mid 8$, so $n_5 = 1$.

ii) $p = 7$: $n_7 \equiv 1 \pmod 7$ and $n_7 \mid 8$, so $n_7 = 1$ or 8.

(b) If $|G| = 40$, then $p = 5$, and the Sylow 5–subgroup (of order 5) is unique, hence normal in *G*. It is a proper normal subgroup (order < 40), so *G* cannot be simple.

If $|G| = 56$, then $p = 7$, and there may be 1 or 8 Sylow 7–subgroups which are cyclic and of order 7. If $n_7 = 1$ then the 7–subgroup is a proper normal subgroup of *G*, so *G* is not simple.

If $n_7 = 8$, the subgroups are disjoint (prime order) and each contains 6 elements of order 7. Therefore *G* has at least 48 elements of order 7.

This leaves only 8 elements (including the identity), sufficient for exactly one Sylow 2–subgroup, which has order 8. Thus the Sylow 2–subgroup must be unique, hence normal. Therefore, again, *G* cannot be simple.

(e) If $p > 7$ then $p \geq 11$, and $n_p \equiv 1 \pmod p$ and $n_p \mid 8$. We must have $n_p = 1$ so the Sylow *p*–subgroup is unique, hence normal in *G*, with order $p < |G|$, and *G* cannot be simple.

Theorem 3.1 (Summary of the Sylow results) HB p48

Definition of simple group HB p49