M336 2001 Exam Solutions

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Health Warning: This is NOT official OU material. The solutions show how I personally would have tackled the paper. They have not been verified, and it is unlikely that they would agree totally with the Exam Board's answers. (Let me know if you spot any errors)

Part I (55 marks - 90 minutes)

Question 1.

(a) Tile type: [4,8,8]
Vertex types: (3,3,3,3) (3,3,3,3,3,3,3,3)
Tile uniform, but not vertex uniform.

(b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is orthogonal, since

$(0, 1).(0, 1) = (-1, 0).(-1, 0) = 1$ and $(0, 1).(-1, 0) = 0$

(a) HB 9-10
Don't bother to memorise which brackets to use.

(b) HB p9 Definition of orthogonal matrix

Question 2.

(a) $s \wedge r = srs^{-1} = (sr)s^{-1} = (r^4 s)s^{-1} = r^4 (ss^{-1}) = r^4$

(b) $r^m \wedge r = r^m r r^{-m} = r^{m+1-m} = r$

$r^m s \wedge r = (r^m s)\, r\, (r^m s)^{-1} = r^m (sr) s^{-1} r^{-m} = r^m (r^4 s) s^{-1} r^{-m} = r^m r^4 r^{-m} = r^4$

So Stab(r) = {e, r, r^2, r^3, r^4)

(b) Easier way (since you've already done $s \wedge r$):
$r^m s \wedge r = r^m \wedge (s \wedge r)$
$= r^m \wedge r^4 = r^m r^4 r^{-m} = r^4$

Question 3.

(a) $\mathscr{F}$: v? no; h? no; g? no; r? yes; so Type 5

(b) $\mathscr{G}$: v? yes; h? no; g? yes; so Type 7

HB p14 Algorithm

Question 4.

a) $\phi : G \to Z$, $x^n \mapsto n$ $\qquad \psi : G \to Z$, $x^n \mapsto -n$

b) $\psi : G \to Z$ is an isomorphism, so $\psi^{-1} : Z \to G$ is an isomorphism, where $\psi^{-1}(n) = x^{-n}$. Then $\psi^{-1}\phi(x^n) = \psi^{-1}(n) = x^{-n}$, and $\psi^{-1} \circ \phi : G \to G$ is an isomorphism mapping g to g^{-1} for all $g \in G$.

b) Theorem 4.1 HB p17 (to justify "ψ^{-1} is an isormorphism").
I couldn't find a theorem in the handbook to justify "$\psi^{-1} \circ \phi$ is an isormorphism"

Question 5.

(a) $G \cong C_6$, so $|G| = 6$.

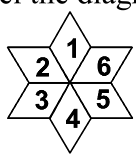
g	**cycle example**	**cs(g)**	**number**
e	(1)(2)(3)(4)(5)(6)	x_1^6	1
r, r^5	(123456)	x_6^1	2
r^2, r^4	(135)(246)	x_3^2	2
r^3	(14)(25)(36)	x_2^3	1

Cycle index: $\frac{1}{6}\left(x_1^6 + 2x_6^1 + 2x_3^2 + x_2^3\right)$

Total with 2 colours: $\frac{1}{6}\left(2^6 + 2 \cdot 2^1 + 2 \cdot 2^2 + 2^3\right) = 14$

(b) Inserting one pink petal destroys the rotational symmetry.
So there are $2^6 = 64$ essentially different colourings.

(a) Label the diagram:



HB p27 Cycle Index Th'm.

(b) Put the pink petal at the top and colour each of the other petals independantly,

Question 6.

(a) Maximum order of an element in Z_{12} is 12, and of an element in Z_{18} is 18. So maximum order of an element in $Z_{12} \times Z_{18}$ is lcm(12, 18) = 36.

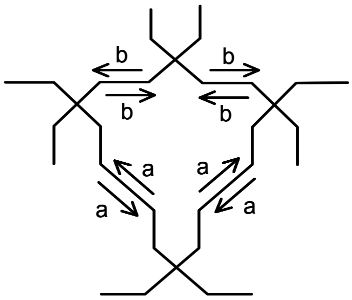
(b) Order 3 elements in Z_{12} are 4 and 8, and in Z_{18} are 6 and 12 so order 3 elements in $Z_{12} \times Z_{18}$ are:
(0,6), (0,12), (4, 0), (4, 6), (4, 12), (8, 0), (8, 6), (8, 12)

a) Uses Result 2.6 HB p 22

b) $|(a, b)|$ has order 3 if lcm($|a|$, $|b|$) = 3, so must have:
$|a| = 3$ and $|b| = 1$ or 3, or
$|b| = 3$ and $|a| = 1$ or 3

Question 7.

(a)



(b) $a^{\rightarrow}$ $b^{\rightarrow}$ $b^{\leftarrow}$ $a^{\leftarrow}$
$a^{\leftarrow}$ $b^{\leftarrow}$ $b^{\rightarrow}$ $a^{\rightarrow}$

(c) Edge labelled a: same and oppositely directed, so edge stabliliser is $\{e, r\}$

(c) Th'm 5.2, HB p30

Question 8.

$$(a)\ (b)\quad \begin{bmatrix} 2 & -2 & 0 & -4 \\ 0 & 3 & 3 & 9 \\ 4 & 2 & 6 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 & 0 & -4 \\ 0 & 3 & 3 & 9 \\ 0 & 6 & 6 & 18 \end{bmatrix} \quad R_3 \rightarrow R_3 - 2R_1$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 9 \\ 0 & 6 & 0 & 18 \end{bmatrix} \quad \begin{matrix} C_2 \rightarrow C_2 + C_1 \\ C_3 \rightarrow C_3 - C_2 \\ C_4 \rightarrow C_4 + 2C_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} C_4 \rightarrow C_4 - 3C_2 \\ R_3 \rightarrow R_3 - 2R_2 \end{matrix}$$

(c) Rank is 2.

(b) **Watchpoint:**
Show your working. It's easy to make arithmetic errors, so you need to show what you were trying to do, even if you get it wrong.

Question 9.

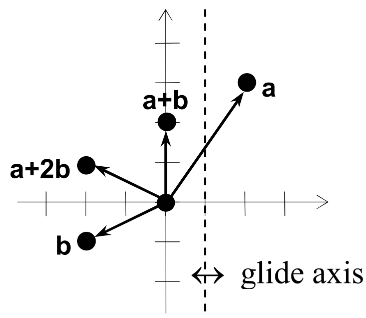
(a) Reduced basis: {(0, 2), (–2, –1)} (= {**a**+**b**, **b**})

(b) L has a basis of equal length vectors, {**b**, **a**+2**b**} = {(–2, –1), (–2, 1)}, and the angle between them is not $\pm\pi/2$ or $\pm\pi/3$, so L is *rhombic*.

Alternative: use the diagram in *P&S*, Section 4.3:
Reduced basis {**a'**, **b**}, **a'** = (0, 2). $|\mathbf{a'}| = 2$, $|\mathbf{b}| = \sqrt{5}$, $|\mathbf{a'}.\mathbf{b}| = 2 = \frac{1}{2}|\mathbf{a'}|^2$

(c) $g = t[(0, 3)]\ q[(1, 0), \pi/2]$ (*Alternative*: $t[(2, 0)]\ q[(0, {}^3/_2), 0]$)

Picture, with lots of vertices.



(b) Classification of lattices: Th'm 5.3, HB p40.

Question 10.

We have $|H| = |K| = |G|/n$, and these are the only subgroups of this order.
For each $g \in G$, gHg^{-1} is a subgroup of order $|H|$, so $gHg^{-1} = H$ or K.

If H is normal in G, then $gHg^{-1} = H$ for all $g \in G$, and similarly for K.

If H is not normal, then there is a $g \in G$ such that $gHg^{-1} \neq H$, so $gHg^{-1} = K$.
Then $g^{-1}Kg = H$, which shows that K is also not normal.

A similar argument shows that if K is not normal, then neither is H.

Therefore H is normal if and only if K is normal.

Note 1:
A margin note in GR5 (p19) says you know that gHg^{-1} is a subgroup of order $|H|$.
I couldn't find a theorem in the handbook to justify it.

Note 2:
The statement is of the form $p \Leftrightarrow q$, where p is 'H is normal' and q is 'K is normal'.

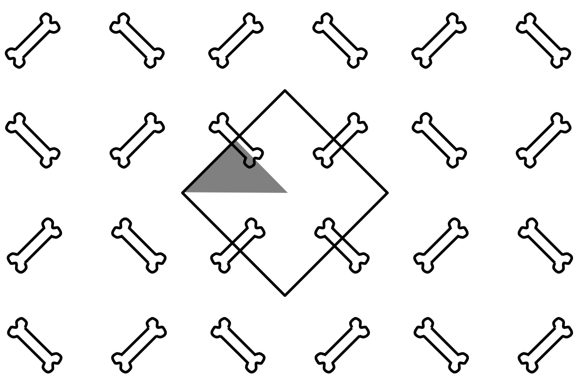
We usually prove both $q \Rightarrow p$ *and* $p \Rightarrow q$.

Here, I've proved the **contrapositive** of these, ie $\neg p \Rightarrow \neg q$ and $\neg q \Rightarrow \neg p$.
(See *P&S* section 2.4).

Question 11.

(a) Highest order of rotation? 4
Any reflections? Yes
4 directions? Yes
Therefore type p4mm.

(b)



(b) The point group of type p4mm is D_4 (HB p46).

Since $|D_4| = 8$, the area of the generating region is $^1/_8 \times$ the area of the basic parallelogram. [Unit GE4, p8 final paragraph,
and P&S HB supplement]

Question 12

(a) $G \cong D_6$

Axis through:	Rotation through:	Label	Number
centres of hexagons	$\pm \pi/3$	r, r^5	2
	$\pm 2\pi/3$	r^2, r^4	2
	π	r^3	1
centres of rectangles	π	$s,\ r^2s,\ r^4s$	3
edges of rectangles	π	$rs,\ r^3s, r^5s$	3

(b)

g	Cycle type	cs(g)	Number
e	$(*)(*)\ldots(*)(*)$	x_1^{18}	1
r, r^5	$(******)(******)(******)$	x_6^3	2
r^2, r^4	$(***)(***)(***)(***)(***)(***)$	x_3^6	2
r^3	$(**)(**)\ldots(**)(**)$	x_2^9	1
$s,\ r^2s,\ r^4s$	$(*)(*)(**)(**)\ldots..(**)(**)$	$x_1^2x_2^8$	3
$rs,\ r^3s, r^5s$	$(**)(**)\ldots(**)(**)$	x_2^9	3

Cycle index: $^1/_{12}(x_1^{18} + 2x_6^3 + 2x_3^6 + 4x_2^9 + 3x_1^2x_2^8)$

(c) With m types of wood: $^1/_{12}(m^{18} + 2m^3 + 2m^6 + 4m^9 + 3m^{10})$

(d) Let $S \subseteq X$ be the set of colourings of boxes with the special property, ie on each surface, opposite faces are made of the same wood. (X is the set of *all* colourings.)

The given θ maps opposite triangles/rectangles to each other, so every element of S is fixed by θ, and θ fixes only those elements whose opposite shapes are alike.

Therefore $S = \text{Fix}(\theta)$.

For each $g \in G$, let $\text{Fix}_S(g)$ be the elements of S fixed by g.
The number of equivalence classes is the number of orbits of elements of S, which, by the Counting Lemma, is $\frac{1}{12}\sum_{g\in G}|\text{Fix}_S(g)|$.

For each $g \in G$, and any $x \in X$, if $x \in \text{Fix}_S(g)$, then $x \in \text{Fix}(g)$ and $x \in S = \text{Fix}(\theta)$, so $x \in \text{Fix}(g) \cap \text{Fix}(\theta)$. Hence $\text{Fix}_S(g) \subseteq \text{Fix}(g) \cap \text{Fix}(\theta)$.

If $x \in \text{Fix}(g) \cap \text{Fix}(\theta)$, then x is fixed by g, and $x \in \text{Fix}(\theta) = S$, so $x \in \text{Fix}_S(g)$, and $\text{Fix}(g) \cap \text{Fix}(\theta) \subseteq \text{Fix}_S(g)$.

Therefore $\text{Fix}_S(g) = \text{Fix}(g) \cap \text{Fix}(\theta)$.

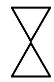
Hence, $\frac{1}{12}\sum_{g\in G}|\text{Fix}_S(g)| = \frac{1}{12}\sum_{g\in G}|\text{Fix}(g) \cap \text{Fix}(\theta)|$.

Continued on next page

Question 12 continued:

(d) cont:

A box with the special property is made up of 9 objects: 3 pairs of triangles on each hexagonal surface, and 3 pairs of opposite rectangles. It is these objects that constitute $\mathrm{Fix}_S(g) = \mathrm{Fix}(g) \cap \mathrm{Fix}(\theta)$.

Pair of triangles: 

g	$\lvert\mathrm{Fix}_S(g)\rvert$	Why?	Number
e, θ	m^9	Each object coloured independently	2
rotations through $\pm\pi/3$, $\pm 2\pi/3$	m^3	All objects on the same surface must be the same colour (m choices for each of the 3 surfaces).	4
rotations that turn the box over	m^5	m choices for each of 3 pairs of triangles on one hexagonal surface, which determines the other surface. 2 pairs of rectangles must be the same colour (m choices), 1 pair coloured independently (m choices).	6

So number of equivalence classes = $^1/_{12}(2m^9 + 4m^3 + 6m^5)$.

Equivalent cycle symbols:

$x_1^{\,9}$

$x_3^{\,3}$

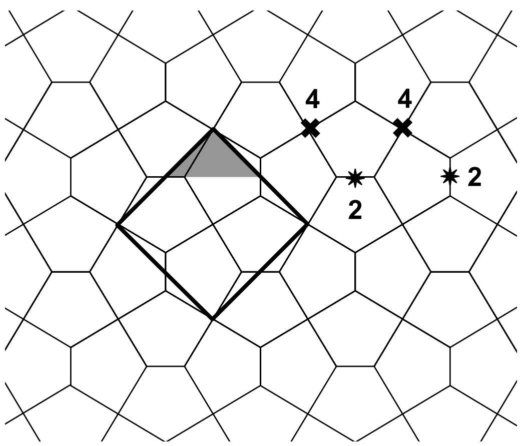
$x_1 x_2^{\,4}$

(Each pair of triangles on the top maps to a pair on the bottom, giving 3 2-cycles. The 4th 2-cycle comes from the rectangles)

Question 13.

(a) Each tile has 4 long edges and 1 short edge. A tile cannot be rotated about any interior point, since there is nowhere for the short edge to go.

(b)
(c)



(d) highest order? 4
reflections? yes
4 directions? no therefore *p*4*gm*

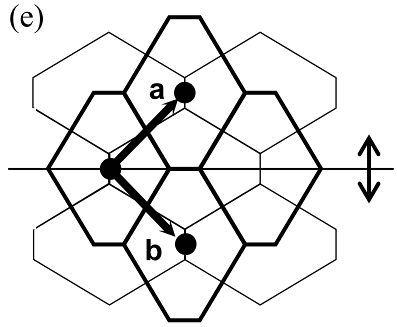
(e) Separating the two orientations would remove the order 4 rotations.

highest order? 2
reflections? yes
2 directions? yes
type? rhombic, therefore *c*2*mm*

Check: From part (d) you know that the point group of *p*4*gm* is D_4 with order 8.
So the area of the generating region is $^1/_8$ × the area of the basic parallelogram.

(d) Algorithm HB p 45–46

(e)



Definition: HB p43
For this basis, reflection in this axis gives:
$q(\mathbf{a}) = \mathbf{b}$ and $q(\mathbf{b}) = \mathbf{a}$, so the reflection is rhombic.

Question 14.

(a) (i) **b**.**c** = (0,0,2).(0,4,0) = 0, so **b** and **c** are orthogonal, and $\|\mathbf{b}\| \neq \|\mathbf{c}\|$.

Set **a' = b, b' = c, c' = a**. Transition matrix from {**a',b',c'**} to {**a,b,c**} is

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ All entries are integers, and determinant equals 1.

Therefore L(**a',b',c'**) = L(**a,b,c**) [Th'm 1.2], and L(**a',b'**) is rectangular.

(ii) **c'** = (3,0,0) + 0**a'** + ½**b'**, where (3,0,0) is orthogonal to **a'** and **b'**, so offset = (0, ½), and L is base-centred orthorhombic.

(b) (i) Let {**a, b**} = {(1, 1, 0), (1, −1, 0)}, a reduced basis for the plane lattice. If **c** = (1, 0, 1), then {**a, b, c**} is a basis for the space lattice.

(ii) L(**a**,**b**) is square, since **a.b** = 0, and $\|\mathbf{a}\| = \|\mathbf{b}\|$ (= √2).

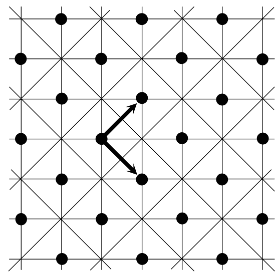
c = (0,0,1) + ½**a** + ½**b**, where (0,0,1) is orthogonal to **a** and **b**, so offset = (½, ½). Vertical separation = 1 = $^1/_{\sqrt{2}}\|\mathbf{a}\|$.

Therefore, L(**a,b,c**) is face-centred cubic.

(a) (ii) HB p56 Result 3.2

Note: L(**a, b**) is also rectangular, but the change in basis makes the rest of the question much easier.

(b) (i) Draw the associated lattice points to find a reduced basis



Part II B (Groups)

Question 15

(a) (i) $Ne = \{e, r^2, r^4\}$ $\quad Ns = \{s, r^2s, r^4s\}$

$Nr = \{r, r^3, r^5\}$ $\quad Nrs = \{rs, r^3s, r^5s\}$

(ii) $\{e\}$ and $\{r^2, r^4\}$ are both conjugacy classes in D_6. Therefore N is a union of conjugacy classes, so N is normal in D_6.

(iii) $NeK = \{e, r^2, r^4\} \times \{e, r^3, rs, r^4s\}$

$= \{e, r, r^2, r^3, r^4, r^5, s, rs, r^2s, r^3s, r^4s, r^5s\} = D_6$

$NrK = \{r, r^3, r^5\} \times \{e, r^3, rs, r^4s\}$

$= \{e, r, r^2, r^3, r^4, r^5, s, rs, r^2s, r^3s, r^4s, r^5s\} = D_6$

Similarly, $NgK = D_6$ for all $g \in D_6$.

(iv) There is only one double coset.

(b) (i) If $g \in Ng_1K$, then $g = n_1g_1k_1$, and if $g \in Ng_2K$, then $g = n_2g_2k_2$, where $n_1, n_2 \in N$, and $k_1, k_2 \in K$.

Therefore $n_1g_1k_1 = n_2g_2k_2$

$\Rightarrow \quad g_1 = n_1^{-1}n_2g_2k_2k_1^{-1} = n'g_2k'$ for some $n' \in N$, $k' \in K$.

Then $Ng_1K = \{ng_1k : n \in N, k \in K\}$

$= \{n(n'g_2k')k : n \in N, k \in K\}$

$= \{(nn')g_2(k'k) : n \in N, k \in K\}$

$= Ng_2K$ (nn' and $k'k$ run through N and K as n and k do)

(ii) Since N and K are subgroups, $e \in N$ and $e \in K$.

Then $g = ege \in NgK$ for each $g \in G$.

(iii) Part (i) above shows that double cosets are equal or disjoint, and part (ii) shows that every $g \in G$ is in a double coset.

Therefore the double cosets partition G.

(a) Conjugacy classes in D_6:

$\{e\}$

$\{r, r^5\}$ rotations (± π/3)

$\{r^2, r^4\}$ rotations (± 2π/3)

$\{r^3\}$ rotation (π)

$\{s, r^2s, r^4s\}$ reflections (edges)

$\{rs, r^3s, r^5s\}$ reflections (vertices)

If you didn't remember this, you could have shown that left cosets equal right cosets.

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Question 16.

(a) $|G| = 600 = 2^3 \times 3 \times 5^2$

prime	factors	label
5	5^2	5a
	5×5	5b
3	3	3a
2	2^3	2a
	2×2^2	2b
	$2 \times 2 \times 2$	2c

There are $2 \times 1 \times 3 = 6$ Abelian groups of order 600.

label	*p*-primary form	canonical form
2a3a5a:	$Z_8 \times Z_3 \times Z_{25}$	Z_{600}
2b3a5a:	$(Z_2 \times Z_4) \times Z_3 \times Z_{25}$	$Z_2 \times Z_{300}$
2c3a5a:	$(Z_2 \times Z_2 \times Z_2) \times Z_3 \times Z_{25}$	$Z_2 \times Z_2 \times Z_{150}$
2a3a5b:	$Z_8 \times Z_3 \times (Z_5 \times Z_5)$	$Z_5 \times Z_{120}$
2b3a5b:	$(Z_2 \times Z_4) \times Z_3 \times (Z_5 \times Z_5)$	$Z_{10} \times Z_{60}$
2c3a5b:	$(Z_2 \times Z_2 \times Z_2) \times Z_3 \times (Z_5 \times Z_5)$	$Z_2 \times Z_{10} \times Z_{30}$

(b) The Klein group is isomorphic to $Z_2 \times Z_2$, so p-primary form must contain $Z_2 \times Z_2$. No cyclic subgroups of order 4, so no Z_4 in p-primary form.

Therefore only $Z_2 \times Z_2 \times Z_{150}$ and $Z_2 \times Z_{10} \times Z_{30}$.

(c) $60 = 2^2 \times 3 \times 5$. Let $G \cong A \times B \times C$, where A, B, C are the 2-, 3- and 5-primary components respectively. Then an element of G may be expressed in the form (a, b, c), where $a \in A$, $b \in B$ and $c \in C$.

If $|(a, b, c)| = 60$, then $|a| = 4$, $|b| = 3$, and $|c| = 5$.

Z_8:	2 elements of order 4	Z_3:	2 elements of order 3
$Z_2 \times Z_4$:	4 elements of order 4	Z_{25}:	4 elements of order 5
$Z_2 \times Z_2 \times Z_2$:	no elements of order 4	$Z_5 \times Z_5$:	24 elements of order 5

$Z_8 \times Z_3 \times Z_{25}$	$2 \times 2 \times 4 = 16$ elements of order 60
$(Z_2 \times Z_4) \times Z_3 \times Z_{25}$	$4 \times 2 \times 4 = 32$ elements of order 60
$(Z_2 \times Z_2 \times Z_2) \times Z_3 \times Z_{25}$	no elements of order 60
$Z_8 \times Z_3 \times (Z_5 \times Z_5)$	$2 \times 2 \times 24 = 96$ elements of order 60
$(Z_2 \times Z_4) \times Z_3 \times (Z_5 \times Z_5)$	$4 \times 2 \times 24 = 192$ elements of order 60
$(Z_2 \times Z_2 \times Z_2) \times Z_3 \times (Z_5 \times Z_5)$	has no elements of order 60

(a) Canonical decomposition HB p33 Theorem 3.2
p–primary decomposition HB p38

(c) Counting elements:
Every group has 1 element of order 1.
For prime p:
Z_p :
$p-1$ elements of order p
Z_{p^2} :
exactly one subgroup of order p ($\cong Z_p$), so $p-1$ elements of order p, and $p^2-(p-1)-1 = p^2-p$ elements of order p^2.
$Z_p \times Z_p$:
no elements of order p^2, so p^2-1 elements of order p.

Question 17.	
(a) $\lvert G\rvert = 2 \times 3 \times p$, $p > 5$. Let n_p be the number of Sylow p-subgroups of G. $n_p \equiv 1 \pmod p$ and $n_p \mid 6$, so $n_p = 1$ (since $p \geq 7$) So the Sylow p-subgroup, N, is unique, hence normal in G (Lemma 2.2), and has order $p = \lvert G\rvert/6$. Since $1 < \lvert N\rvert < \lvert G\rvert$, N is a non-trivial, proper normal subgroup of G, and G cannot be simple.	Theorem 3.1 (Summary of the Sylow results) HB p48 Definition of simple group HB p49
(b) $\lvert G\rvert = 2 \times 3 \times 5$ (i) $n_5 \equiv 1 \pmod 5$ and $n_5 \mid 6$, so $n_5 = 1$ or 6 A group of order 5 contains 4 elements of order 5, so if $n_5 = 1$, G has 4 elements of order 5, and if $n_5 = 6$, G has $6 \times 4 = 24$ elements of order 5, since the intersection of any two of the subgroups is the identity. (ii) $n_3 \equiv 1 \pmod 3$ and $n_3 \mid 10$, so $n_3 = 1$ or 10 A group of order 3 contains 2 elements of order 3, so if $n_3 = 1$, G has 2 elements of order 3, and if $n_3 = 10$, G has $10 \times 2 = 20$ elements of order 3, since the intersection of any two of the subgroups is the identity. (iii) If both $n_5 = 6$ and $n_3 = 10$, then G would contain 24 elements of order 5 and 20 elements of order 3, i.e. at least 44 non-identity elements. This is a contradiction, since $\lvert G\rvert = 30$, so at least one of n_5 and n_3 must equal 1. Hence G contains a unique, hence normal Sylow p-subgroup, which is proper and non-trivial, so G cannot be simple	