M336 2002 Exam Solutions

Shirleen Stibbe (01372 275540) ss65@tutor.open.ac.uk

<u>Health Warning</u>: **This is NOT official OU material. The solutions show how I personally would have tackled the paper. They have not been verified, and it is unlikely that they would agree totally with the Exam Board's answers.** (Let me know if you spot any errors)

Part I (55 marks - 90 minutes)

Question 1.

(a) Tile type: [4,6,6] [6,6,6] [6,6,8]
Vertex types: (3,3,3,3) (3,3,3,3,3,3,) (3,3,3,3,3,3,3,3)

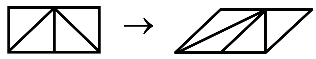
(b) Tiles within a square rotate into each other through angle π, so will remain congruent under an affine transformation.

Tiles in adjacent squares are at angle $\pi/2$, so may not remain congruent under an affine transformation.

Therefore, only two non-congruent tile shapes can occur under an affine transformation.

(a) HB 9-10
Don't bother to memorise which brackets to use.
(b) HB p4 Lemma 4.1
Affine congruency lemma

eg $(x, y) \to (x + y, y)$



Question 2.

(a) $(r^3s)(r^5s) = r^3(sr^5)s = r^3(r^{-5}s)s = r^3(r^4s)s = r^7$ since $s^2 = e$.

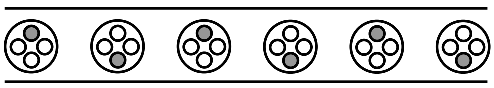
(b) i) $H = \{e, r^3, r^6, r^2s, r^5s, r^8s\}$
ii) $|H| = 6$

(a) Result 3.3.2, HB Supp
$sr^m = r^{-m}s$

(b) Result 3.9.1, HB Supp
with $N = \langle r^3\rangle$, $q = r^2s$

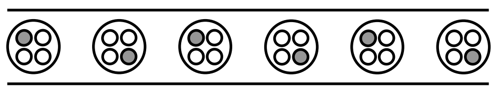
Question 3.

(a) $\mathscr{F}$:



Type 7: v ✓ h ✗ g ✓

(b) $\mathscr{G}$:



Type 5: v ✗ h ✗ g ✗ r ✓

HB p14 (Algorithm)
(a) Kill horizontal reflection, keep glide.
(b) Kill all reflections and glides, keep rotation.

Question 4.

<u>Closure:</u> Let $x = gh_1g^{-1}$, $y = gh_2g^{-1} \in gHg^{-1}$.

Then $xy = gh_1g^{-1}gh_2g^{-1} = gh_1h_2g^{-1} \in gHg^{-1}$, since $h_1h_2 \in H$ (closed).

<u>Identity</u>: $e = geg^{-1} \in gHg^{-1}$.

<u>Inverses</u>: Let $x = ghg^{-1} \in gHg^{-1}$.

Then $x^{-1} = (ghg^{-1})^{-1} = (g^{-1})^{-1}h^{-1}g^{-1} = gh^{-1}g^{-1} \in gHg^{-1}$, since $h^{-1} \in H$.

Therefore gHg^{-1} is a subgroup of G.

Question 5.

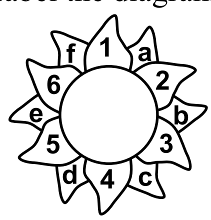
(a) $G \cong C_6 = \langle r: r^6 = e\rangle$ so $|G| = 6$.

g	cycle type	cs(g)	number
e	(1) … (6)(a) … (f)	x_1^{12}	1
r, r^5	(123456)(abcdef)	x_6^2	2
r^2, r^4	(135)(246)(ace)(bdf)	x_3^4	2
r^3	(14)(25)(36)(ad)(be)(cf)	x_2^6	1

Cycle index: $\frac{1}{6}\left(x_1^{12} + 2x_6^2 + 2x_3^4 + x_2^6\right)$

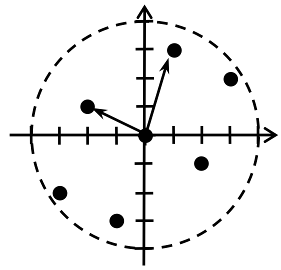
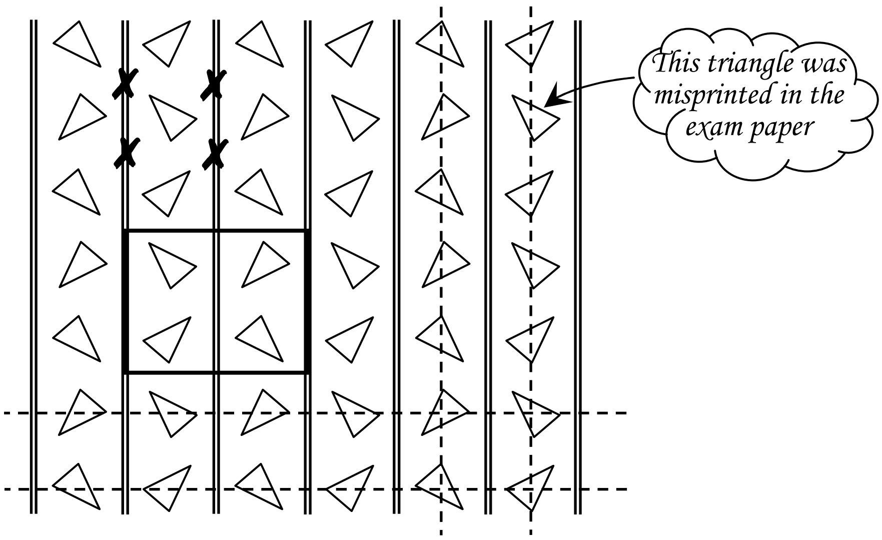
(b) Total with 2 colours: $\frac{1}{6}\left(2^{12} + 2\cdot 2^2 + 2\cdot 2^4 + 2^6\right) = 700$

(a) Label the diagram:



NB: there are 12 objects in the set that G is acting upon, so cs(g) = x_a^b, where $a \times b = 12$

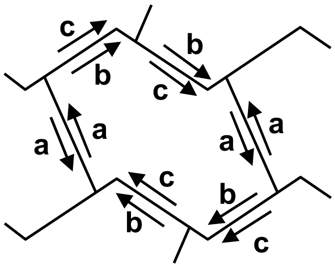
HB p27 Cycle Index Th'm.

Question 6. (a) $\{(0, 0), (1, 1), (0, 2), (1, 3)\}$. (b) $\{(0, 0), (1, 0), (0, 2), (1, 2)\}$.	(a) $\langle(1, 1)\rangle$ (b) 3 elements of order 2
Question 7. (a) $\mathcal{T}_A \cong \mathcal{T}_B$ (both isomorphic to $\mathcal{R}_A$) (b) i) $n_t(\mathcal{T}_A) = 4$, $n_v(\mathcal{T}_A) = 4$ ii) $n_e(\mathcal{T}_A) = n_t(\mathcal{T}_A) + n_v(\mathcal{T}_A)$ (Euler's equation), so $n_e(\mathcal{T}_A) = 8$	(a) Th'm 1.2, HB p29 (b) Euler's equation: Th'm 2.2, HB p29 Or you could have used the Edge orbit th'm (3.1)
Question 8. $A \cong Z_4 \times Z_5 \times (Z_2 \times Z_3) \times (Z_2 \times Z_5 \times Z_7) \cong Z_2 \times Z_{10} \times Z_{420}$ Torsion coefficients are 2, 10 and 420 $B \cong Z_3 \times Z_7 \times (Z_4 \times Z_5) \times (Z_4 \times Z_5) \cong Z_{20} \times Z_{420}$ Torsion coefficients are 20 and 420 A and B are not isomorphic, since their torsion coefficients differ.	Th'm 3.2 HB p33
Question 9. (a) (0, 0), (−2, 1), (2, −1), (1, 3), (−1, −3), (3, 2), (−3, −2) (b) Reduced basis: $\{\mathbf{a'}, \mathbf{b'}\} = \{(-2, 1), (1, 3)\}$ [*NB: smallest first*] (c) L is a parallelogram lattice. *Note*: $\|\mathbf{a'}\| = \sqrt{5}$, $\|\mathbf{b'}\| = \sqrt{10}$, $\mathbf{a'} \cdot \mathbf{b'} = 1$ ($\neq 0$ or $\frac{1}{2}\|\mathbf{a'}\|^2$)	 (c) HB Supp p3
Question 10. $\|G\| = 63$, and $Z(G)$ is a normal subgroup of G, so $\|Z(G)\|$ divides 63. Hence $\|Z(G)\| \in \{1, 3, 7, 9, 21, 63\}$. $\|Z(G)\| \neq 63$, since G is non–Abelian. If $\|Z(G)\| \in \{9, 21\}$, then $\|G / Z(G)\| \in \{7, 3\}$, and the quotient group would be cyclic (prime order). This implies that G is Abelian – a contradiction. Therefore $\|Z(G)\| \leq 7$.	Result 4.2, HB p37 Th'm 4.3 HB p37
Question 11. 	HB Supp p3 to draw the basic parallelogram.

Part II A (Geometry)

Question 12

(a)



(b) $\mathbf{a}^{\rightarrow}\ \mathbf{b}^{\rightarrow}\ \mathbf{c}^{\rightarrow}\ \mathbf{a}^{\rightarrow}\ \mathbf{b}^{\rightarrow}\ \mathbf{c}^{\rightarrow}$

$\mathbf{a}^{\leftarrow}\ \mathbf{c}^{\rightarrow}\ \mathbf{b}^{\rightarrow}\ \mathbf{a}^{\leftarrow}\ \mathbf{c}^{\rightarrow}\ \mathbf{b}^{\rightarrow}$

(c) 3 edge side orbits

(d) 2 edge orbits

(e) $n_t(\mathcal{T}) = 2$, $n_e(\mathcal{T}) = 6$, $n_v(\mathcal{T}) = 4$

(f) $\{e, r\}$ [labels the same and oppositely directed]

(g) No, because wallpaper type *p*2*gg* has no reflections, whereas there must be a reflection axis bisecting an edge with stabiliser $\{e, v\}$ [the label is undirected, i.e. the edge reflects into itself].

(c) 3 different letters needed in a tile, so 3 different edge side orbits

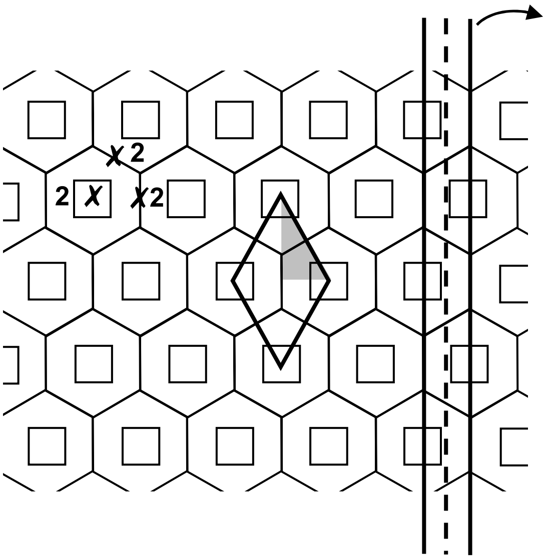
(d) **b** ↔ **c**, so **b** and **c** are in the same edge orbit

Edge stabilisers:
Th'm 5.2 (b), HB p30

Classifying wallpapers
HB p45

Question 13.

(a) (c)



Not asked for – only used to check type of reflections

(b) Highest order? 2
Reflections? yes
2 directions? yes
type? rhombic
Therefore type *c*2*mm*

(d) Rotation r of order 4 is through angle $\pi/2$.
Rotation r' of order 6 is through angle $\pi/3$.
The composition rr' is a rotation through $(\pi/2 + \pi/3) = \pi/6$, which has order 12. This would violate the chrystallographic restriction. So no wallpaper pattern can have rotations both of order 4 and of order 6.

(b) Algorithm HB p 45–46
Note:
Alternating glide and reflection axes, so you know the reflections are rhombic
(Th'm 2.7, HB p44)

(c) Point group of *c*2*mm* is D_2 (HB p46), of order 4.
So area of the generating region is $^1/_4 \times$ area of the basic parallelogram.

Basic parallelogram and generating region:
HB Supp p3 & 4

(d) *Note*: chrystallographic restriction applies to wallpapers via the associated lattice.

Question 14.

(a) (i) $\|\mathbf{a}\| = \|\mathbf{b}\| = 3$, $\mathbf{a}\cdot\mathbf{b} = 3$ ($\neq 0$ or $\frac{1}{2}\|\mathbf{a}\|^2$), so $L(\mathbf{a}, \mathbf{b})$ is <u>rhombic</u>.

(ii) $\mathbf{c} = (0, 0, 2) + \frac{1}{2}\mathbf{a} + (-1 + \frac{1}{2})\mathbf{b}$, where $(0, 0, 2)$ is orthogonal to **a** and **b**, so offset = $(\frac{1}{2}, \frac{1}{2})$, and L is <u>face-centred orthorhombic</u>.

Continued on next page

HB Supp p3 (2-d lattices) and p4 (Bravais lattices).
Note: L is rhombic, and $\|\mathbf{a}\| = \|\mathbf{b}\|$, so $\{\mathbf{a},\mathbf{b}\}$ is a suitable basis

(c) Result 3.2, HB p56.

Question 14 (continued).

(b) i) $G^+ \cong S_4$, $|G^+| = 24$.
The truncated cube contains 6 octagons and 8 triangles.

Rotation axes	Angle	No	Detail	$\|Fix(g)\|$
identity	0	1	all coloured independently	m^6n^8
centres of octagons	$\pm \pi/2$	6	octagons: 4 the same, 2 independent triangles: 2 sets of 4 the same	m^3n^2
centres of octagons	π	3	octagons: 2 opposite pairs the same, 2 independent triangles: opposite pairs the same	m^4n^4
centres of trianges	$\pm 2\pi/3$	8	octagons: 2 sets of 3 the same triangles: 2 sets of 3 the same, 2 independent	m^2n^4
centres of edges joining 2 octagons	π	6	octagons: opposite pairs the same triangles: opposite pairs the same	m^3n^4

ii) Hence the number of rotational equivalence classes is
$^1/_{24}(m^6n^8 + 6m^3n^2 + 3m^4n^4 + 8m^2n^4 + 6m^3n^4)$

(b) The direct symmetry group of the truncated cube is exactly the same as the direct symmetry group of the cube:

octagons ≡ faces
triangles ≡ vertices

Note: The detail isn't actually necessary for full marks.

I put it in to show what I thought I was doing.

Probably a good idea for this kind of question – it could gain you some brownie points if you go wrong somewhere along the line.

Part II B (Groups)

Question 15

(a) Closure: Let $h, h' \in H$, so that $h = g^2$, $h' = g'^2$ for some $g, g' \in G$.
Then $hh' = g^2g'^2 = (gg')^2$ (since G is Abeliean)
So $hh' \in H$, since $gg' \in G$ (by closure)
Identity: $e^2 = e \in H$, since $e \in G$.
Inverse: Let $h = g^2 \in H$. Then $h^{-1} = (g^2)^{-1} = (g^{-1})^2 \in H$, since $g^{-1} \in G$.
Hence H is a subgroup of G.

(b) H is a subgroup of G, so $|H| \le |G|$.
Suppose $|H| < |G|$, so that for some $x, y \in G$, $x \ne y$, but $x^2 = y^2$.
Then $(xy^{-1})^2 = x^2y^{-2} = y^2y^{-2} = e$ (since $x^2 = y^2$).
Since $x \ne y$, $xy^{-1} \ne e$ so $|xy^{-1}| = 2$.
But $|G|$ is odd, and $|xy^{-1}|$ divides $|G|$, so it cannot be even – a contradiction.
Therefore $|H| = |G|$, and so $H = G$.

(c) i) $G = Z_2 \times Z_2 \times Z_2 \times Z_2$, $H = \{(0, 0, 0, 0)\}$
ii) $G = Z_4 \times Z_4$, $H = \{(0, 0), (2, 0), (0, 2), (2, 2)\}$
iii) $G = Z_{16}$, $H = \{0, 2, 4, 6, 8, 10, 12, 14\}$

(a) *Note*:
For a non-Abelian group, $(gg')^2 = gg'gg' \ne ggg'g'$.

(b) Alternative method:

Let ϕ: $G \to H$, $\phi(g) = g^2$.

Then show that ϕ is an isomorphism by proving the homomorphism property [easy, since G is Abelian], onto [also easy] and one-one by:

$\text{Ker}(\phi) = \{g \in G: \phi(g) = e\}$
$= \{g \in G: g^2 = e\}$
$= \{e\}$ since $|g|$ is odd.

Question 16.

(a) $|G| = 3600 = 2^4 \times 3^2 \times 5^2$

prime	factors
5	5^2
	5×5
3	3^2
	3×3

prime	factors
2	2^4
	2×2^3
	$2^2 \times 2^2$
	$2 \times 2 \times 2^2$
	$2 \times 2 \times 2 \times 2$

There are $2 \times 2 \times 5 = 20$ Abelian groups of order 3600.

(b) (i) & (ii)

$$\begin{bmatrix} 2 & 4 & 2 \\ 6 & 15 & 12 \\ 6 & 27 & 636 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 6 \\ 0 & 15 & 630 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\to \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 15 & 630 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$\to \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 15 & 600 \end{bmatrix} \quad C_3 \to C_3 - 2C_2$$

$$\to \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 600 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

(iii) $A \cong Z_2 \times Z_3 \times Z_{600} \cong Z_2 \times Z_3 \times (Z_8 \times Z_3 \times Z_{25})$

$\cong (Z_2 \times Z_8) \times (Z_3 \times Z_3) \times Z_{25}$ (p–primary)

$\cong Z_6 \times Z_{600}$ (canonical)

(c)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 300 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 300 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 100 \end{bmatrix} \quad \begin{bmatrix} 4 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 75 \end{bmatrix}$$

NB: It's too easy to make arithmetic errors in this type of question. Show all your working, so at least it's clear what you were trying to do, even if you don't quite make it unscathed to the end.

(b) (iii) NB: order matters for canonical form, e.g. $Z_{600} \times Z_6$ isn't canonical form

(c) And another one for luck

$$\begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

Question 17.	
(a) $\lvert G\rvert = 2^2 \times p^2$, $p > 3$. Let n_p be the number of Sylow p-subgroups of G. $n_p \equiv 1 \pmod p$ and $n_p \mid 4$, so $n_p = 1$ (since $p \geq 5$) So the Sylow p-subgroup, N, is unique, hence normal in G (Lemma 2.2), and has order p^2. Since $1 < \lvert N\rvert < \lvert G\rvert$, N is a non-trivial proper normal subgroup of G, and G cannot be simple.	Theorem 3.1 (Summary of the Sylow results) HB p48 Definition of simple group HB p49
(b) $\lvert G\rvert = 2^2 \times 3^2 = 36$	
(i) $n_3 \equiv 1 \pmod 3$ and $n_3 \mid 4$, so $n_3 = 1$ or 4	
(ii) Suppose G has 4 Sylow 3-subgroups, H_1, H_2, H_3, H_4. The conjugacy action by G on $X = \{H_1, H_2, H_3, H_4\}$ defines a homomorphism $\psi: G \to \Gamma(X)$, and $\xi: \Gamma(X) \to S_4$ is an isomorphism. Then $\phi: G \to S_4$ is the composition $\xi \circ \psi$. If $\lvert \mathrm{Ker}(\phi)\rvert = 36 = \lvert G\rvert$, then $\mathrm{Ker}(\phi) = G$, and $\phi(g)$ is the identity of S_4 for all $g \in G$. This would imply that each H_i is conjugate only to itself, which contradicts Sylow's 3rd Theorem (Sylow p-subgroups are conjugate to each other). Therefore $\lvert \mathrm{Ker}(\phi)\rvert < 36$.	Derived homomorphism HB p50 Homomorphism properties Th'm 4.4, HB p17
(iii) $\mathrm{Im}(\phi)$ is a subgroup of S_4, and $\lvert S_4\rvert = 24$. So $\lvert \mathrm{Im}(\phi)\rvert \neq 36$. Since $\lvert \mathrm{Im}(\phi)\rvert \times \lvert \mathrm{Ker}(\phi)\rvert = \lvert G\rvert = 36$, we must have $\lvert \mathrm{Ker}(\phi)\rvert \neq 1$.	$\lvert \mathrm{Im}(\phi)\rvert \times \lvert \mathrm{Ker}(\phi)\rvert = \lvert G\rvert$ Result 3.5.1, HB Supp p2
(iv) If G has 1 Sylow 3-subgroup (of order 9), then that subgroup is a proper, non-trivial normal subgroup of G. If G has 4 Sylow 3-subgroups then $\mathrm{Ker}(\phi)$ is a proper, non-trivial normal subgroup of G, by parts (ii) and (iii). Hence no group of order 36 can be simple.	