M336 2003 Exam Solutions

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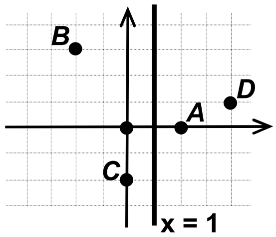
<u>Health Warning</u>: **This is NOT official OU material. The solutions show how I personally would have tackled the paper. They have not been verified, and it is unlikely that they would agree totally with the Exam Board's answers.** (Let me know if you spot any errors)

Part I (55 marks - 90 minutes)

Question 1.



(a) (i) Glide reflection

(ii) $q[(0, -2),\ (1, 0),\ \pi/2]$

(b) $t[(2, -2)]q[\pi/2]$

(a) The clue is that f fixes the line $x = 1$ set-wise. Th^m 5.1 (f), HB p5

(b) f maps the origin to the point $(2, -2)$ HB Supp, Isometry toolkit

Question 2.

(a) $r^2 s = s^3$ (since $r^2 = s^2$), therefore $(r^2 s)^{-1} = s^{-3} = s$ (since $s^4 = r^4 = e$)

(b) $(r^2 s) r (r^2 s)^{-1} = r^2 srs = r^2 r^3 ss = rs^2 = r^3$

(a) Alternative method:

$(r^2 s)^{-1} = s^{-1} r^{-2}$ (***not*** $r^{-2} s^{-1}$)

$= s^3 r^2$ ($s^4 = r^4 = e$)

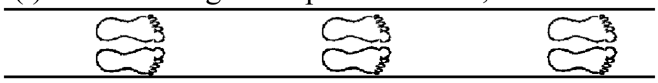
$= (sr^2) r^2$ (since $s^2 = r^2$)

$= s$ (since $r^4 = e$)

Question 3.

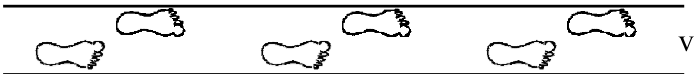
(a) v ✗ h ✗ g ✓ , so Type 4

(b) (i) Translate right footprints 1 stride, introduce horizontal reflection.



v ✗ h ✓ (Type 3)

(ii) Translate right footprints ½ a stride (or any fraction), kill the glide.



v ✗ h ✗ g ✗ r ✗ (Type 1)

HB p14 (Algorithm)

Warning: the 'stride units' given in the question are not to be confused with translation units, which would be 2 strides.

Question 4.

(a) $\phi(e) = e$ (definition) $\qquad \phi(r^2) = [\phi(r)]^2 = a^2 = e$ (morphism property)

$\phi(r^3) = \phi(r^2)\phi(r) = ea = a, \qquad \phi(rs) = \phi(r)\phi(s) = ab = c$

$\phi(r^2 s) = \phi(r^2)\phi(s) = eb = b, \qquad \phi(r^3 s) = \phi(r^3)\phi(s) = ab = c$

(b) $\text{Ker}(\phi) = \{e, r^2\}$, (both elements map to the identity of K_4). By Th^m 4.4 (a) (HB p17), $\text{Ker}(\phi)$ is a normal subgroup of D_4, and has order 2.

(a) Morphism property (HB p19)

Question 5.

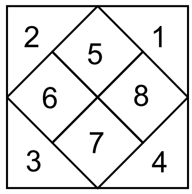
(a) $G \cong D_4$, so $|G| = 8$.

g	**cycle type**	**cs(g)**	**number**
e	(1)(2)(3)(4)(5)(6)(7)(8)	x_1^8	1
r, r^3	(1234)(5678)	x_4^2	2
r^2	(13)(24)(57)(68)	x_2^4	1
$r^n s$	(12)(34)(68)(5)(7)	$x_1^2 x_2^3$	4 ($n = 0, 1, 2, 3$)

Cycle index: $\frac{1}{8}\left(x_1^8 + 2x_4^2 + x_2^4 + 4x_1^2 x_2^3\right)$

(b) Total with 2 colours: $\frac{1}{8}\left(2^8 + 2\cdot 2^2 + 2^4 + 4\cdot 2^2 \cdot 2^3\right) = 51$

(a) Label the diagram:



NB: there are 8 objects in the set that G is acting upon, so if cs(g) = $x_a^b x_c^d$, then $(a \times b) + (c \times d) = 8$

HB Supp p2.

Question 6.

(a) $|G| = 2 \times 3 \times 4 = 24$

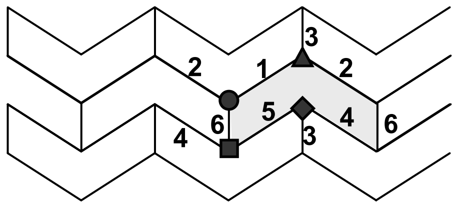
(b) Highest order: $\text{lcm}\{2, 3, 4\} = 12$, $(0,1,1)$, $(1, 1, 1)$, $(0,2,3)$

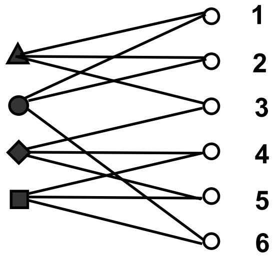
(c) $|0| = 1$ in Z_2, $|1| = 3$ in Z_3, $|2| = 2$ in Z_4, so $|(0,1,2)| = \text{lcm}\{1,3,2\} = 6$

(b) For $|(a,b,c)| = 12$, we need
$a \in Z_2$: 0 or 1 (order 1 or 2),
$b \in Z_3$: 1 or 2 (order 3),
$c \in Z_4$: 1 or 3 (order 4)

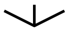
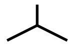
Question 7.

(a)





(b) The two vertical edges incident with any tile are in the same orbit (e.g. in the shaded tile above, the two vertical edges are both in edge-orbit 6). So there are 2 lines going from a tile orbit into one of the edge orbits.

(c) There are two vertex orbits under the full symmetry group $\Gamma(\mathscr{T})$, one containing vertices of this type  and the other of this type .
So the tiling is not vertex-transitive.

(a) 4 vertex orbits, 6 edge orbits, each vertex incident with 3 edges.
Vertex edge diagram HB p32

(c) Vertex transitive HB p32

Question 8.

(a)
(b)

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 12 \\ 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 0 \\ 0 & 3 & 6 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\to \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 3 & 6 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - 3C_1 \end{matrix}$$

$$\to \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

(c) $A \cong Z_2 \times Z_3 \times Z_6 \cong Z_6 \times Z_6$ in canonical form. The torsion coefficients relate to the canonical form, in which each coefficient divides the next.

Th$^{\text{m}}$ 3.2 HB p33

NB: It's a good idea to show exactly what you were trying to do, so that you can get some of the marks even if you commit arithmetic hari kiri doing the actual calculations.

Question 9.

(a) Reduced basis: $\{\mathbf{a}', \mathbf{b}'\} = \{(1, 2), (-2, 2)\}$ [*NB: smallest first*]

$\|\mathbf{a}'\| = \sqrt{5}$, $\|\mathbf{b}'\| = \sqrt{8}$, $\mathbf{a}' \cdot \mathbf{b}' = 2 \neq 0$ or $\frac{1}{2}\|\mathbf{a}'\|^2$,
so L is a parallelogram lattice.

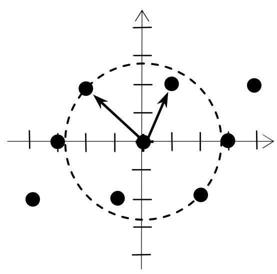
(b) Let $\mathbf{c} = \frac{3}{2}\mathbf{b} = (\frac{3}{2}, 3)$, and $\mathbf{d} = \frac{3}{2}\mathbf{b} - \mathbf{a} = (-\frac{3}{2}, 3)$

$\|\mathbf{c}\| = \|\mathbf{d}\| = \sqrt{\frac{45}{4}}$, $\mathbf{c} \cdot \mathbf{d} = \frac{27}{4} \neq 0$ or $\frac{1}{2}\|\mathbf{c}\|^2$, so $L(\mathbf{c}, \mathbf{d})$ is rhombic.

$\mathbf{a} = \mathbf{c} + (-1)\mathbf{d}$, $\frac{3}{2}\mathbf{b} = \mathbf{c} + 0\mathbf{d}$. The transition matrix from $L(\mathbf{a}, \frac{3}{2}\mathbf{b})$ to $L(\mathbf{c}, \mathbf{d})$ is $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ with integer entries and determinant equal to 1.

So $L(\mathbf{a}, \frac{3}{2}\mathbf{b}) = L(\mathbf{c}, \mathbf{d})$, and $L(\mathbf{a}, \frac{3}{2}\mathbf{b})$ is rhombic.

(a) NB: Draw a picture, with lots of lattice points.



Classification: HB Supp p3

(b) Equal lattices: Th$^{\text{m}}$ 1.2, HB p38

Question 10.

G has order p^n for some integer n, where p is a prime (definition of p-group).

By the Orbit Stabiliser theorem, for each $x \in G$, $|\text{Orb}(x)| \times |\text{Stab}(x)| = |G|$, so $|\text{Orb}(x)|$ divides $|G|$, and $|\text{Orb}(x)| = p^m$ for some $m \leq n$. For single element classes $m = 0$, and $0 < m \leq n$ for orbits with more than one element.

The orbits here are the conjugacy classes. If $m \neq 0$, p divides p^m, and the result follows.

p-group: HB p 38

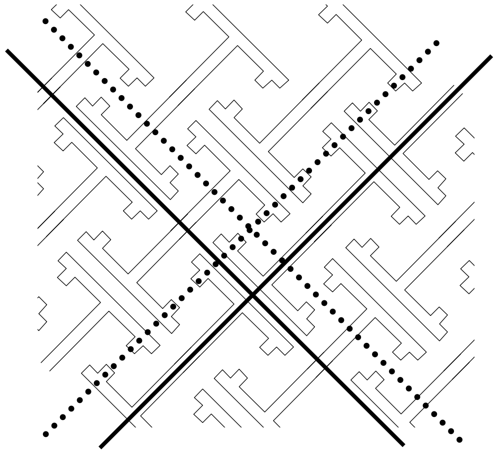
Class equation: Th^{m} 4.1, HB p36

Orbit Stabilizer: Th^{m} 3.2, HB p27

Question 11.

(a) Highest order of rotation is 2

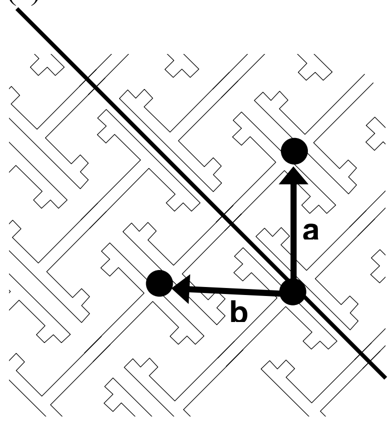
(b)



(c) Symmetries are rhombic

(d) Highest order? 2
Reflections? yes
2 directions? yes
Type of reflection? rhombic
Therefore Type c2mm.

(c)



Reflect these basis vectors in this axis to get:

$q(\mathbf{a}) = \mathbf{b}$, $q(\mathbf{b}) = \mathbf{a}$

Therefore, rhombic (HB p43)

Another clue: glide and reflection axes alternate Th^{m} 2.7, HB p44

(d) Algorithm HB p45

Question 12

(a) The top and bottom joins can be either parallel or perpendicular to each other.

(b) $\boldsymbol{B_1}$: Identity

Rotation r through π about axis through centre of top and bottom

Rotations s, t through π about axes through centres of opposite sides (turning the box over).

rotation	**cycle type**	**cs(g)**	**number**
e	(1)(2)(3)(4)(5)(6)(7)(8)	x_1^8	1
r	(12)(35)(46)(78)	x_2^4	1
s, t	(17)(28)(46)(3)(5)	$x_1^2x_2^3$	2

Cycle index for $\boldsymbol{B_1}$: $\frac{1}{4}\left(x_1^8 + x_2^4 + 2x_1^2x_2^3\right)$

$\boldsymbol{B_2}$: Identity

Rotation r through π about axis through centre of top and bottom

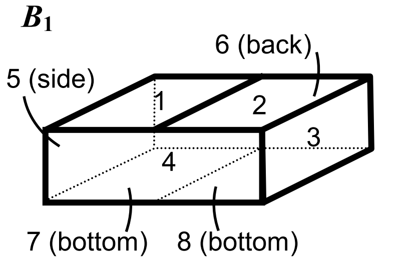
Rotations s, t through π about diagonal axes through centres of opposite short edges (turning the box over).

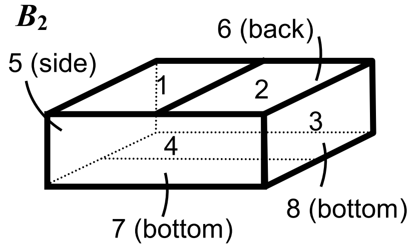
rotation	**cycle type**	**cs(g)**	**number**
e	(1)(2)(3)(4)(5)(6)(7)(8)	x_1^8	1
r	(12)(35)(46)(78)	x_2^4	1
s, t	(18)(27)(34)(56)	x_2^4	2

Cycle index for $\boldsymbol{B_2}$: $\frac{1}{4}\left(x_1^8 + 3x_2^4\right)$

(c) With 3 types of wood, and the two types of boxes, the total number is

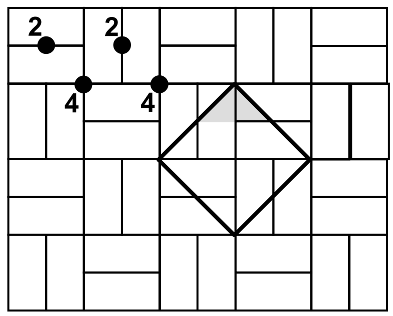
$$\frac{1}{4}\left(3^8 + 3^4 + 2\cdot 3^2\cdot 3^3 + 3^8 + 3\cdot 3^4\right) = 3483$$





Question 13.

(a) & (b)



2 orbits of 2-centres
2 orbits of 4-centres

(b) Point group of $p4gm$ is D_4 (HB p46), of order 8. So area of the generating region is $^1/_8 \times$ area of the basic parallelogram.

Basic parallelogram and generating region: HB Supp pp3 & 4

(c) Algorithm HB p 45–46

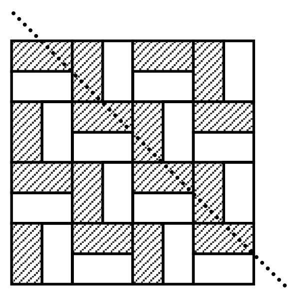
(c) Highest order? 4
Reflections? yes
4 directions? no Therefore type $p4gm$

(d) No new rotations introduced, therefore no rotations of order 3 or 6.

Rotation through π would take a top (red) block to a bottom (blue) block, so no rotations of order 2, and therefore none of order 4 either.

It has no reflections but it does have glides in one diagonal direction, so type pg.

(d)



Question 14.

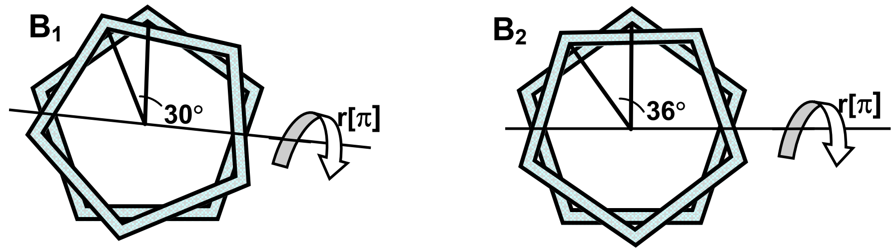
(a) (i) Let $\mathbf{a'} = \mathbf{c}$, $\mathbf{b'} = \mathbf{b}$ and $\mathbf{c'} = \mathbf{a}$. Then $L(\mathbf{a'}, \mathbf{b'})$ is a square lattice, since $\|\mathbf{a'}\| = \|\mathbf{b'}\| = \sqrt{20}$, and $\mathbf{a'}\cdot\mathbf{b'} = 0$.

The transition matrix from $\{\mathbf{a'}, \mathbf{b'}, \mathbf{c'}\}$ to $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is: $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

with integer entries and determinant equal to -1.
Therefore, by Th[m] 1.2, $L(\mathbf{a'}, \mathbf{b'}, \mathbf{c'}) = L(\mathbf{a}, \mathbf{b}, \mathbf{c})$.

(ii) $\mathbf{c'} = (3, 0, 0) + \tfrac{1}{2}\mathbf{a'} + \tfrac{1}{2}\mathbf{b'}$, where $(3, 0, 0)$ is orthogonal to $\mathbf{a'}$ and $\mathbf{b'}$, so offset = $(\tfrac{1}{2}, \tfrac{1}{2})$, and $\|\mathbf{c'}\| = 3 \neq {}^{1}/_{2}\|\mathbf{a'}\|$ or ${}^{1}/_{\sqrt{2}}\|\mathbf{a'}\|$.
Therefore L is <u>body-centred tetragonal.</u>

(b) (i)



Both brooches have order 5 rotations about the centre (without turning over) and 5 rotations of order 2 (turning the brooches over).

So $G_1^+ \cong G_2^+ \cong D_5$

B_1 has no indirect symmetries, so $G_1 \cong G_1^+ \cong D_5$

B_2 has 5 reflections and central inversion, so $G_2 \cong D_5 \times C_2 \cong D_{10}$.

(ii) If the pentagons are of different colours, then B_2 cannot be turned over, so $G_2^+ \cong C_5$. Central inversion is lost, but the 5 reflections remain, so $G_2 \cong D_5$.

(a)
HB Supp p3 (2-d lattices) and p4 (Bravais lattices), and Result 3.2, HB p56.

(b) (ii) Result 5.1 offers either C_{10} or D_5. The 5 reflections are each of order 2, so $G_2 \not\cong C_{10}$. (C_{10} has only 1 element of order 2.).

Each pentagon is fixed independently by D_5.

Part II B (Groups)

Question 15

(a) (i) $eH = \{e, r, r^2\}$ $\quad rsH = \{rs, rsr, rsr^2\}$
$sH = \{s, sr, sr^2\}$ $\quad r^2sH = \{r^2s, r^2sr, r^2sr^2\}$

$eK = \{e, s, rsr^2, r^2sr\}$
$rK = \{r, rs, r^2sr^2, sr\}$
$r^2K = \{r^2, r^2s, sr^2, rsr\}$

(ii) $Hs = \{s, rs, r^2s\} \neq sH$, so H is not normal in G.

$Ke = \{e, s, rsr^2, r^2sr\} = eK$
$Kr = \{r, sr, rs, r^2sr^2\} = rK$
$Kr^2 = \{r^2, sr^2, rsr, r^2s\} = r^2K$
so K is normal in G

(b) (i) K is normal in G, so $Kg = gK$, therefore $KgK = gKK = gK$

(ii) $HeH = HH = H = \{e, r, r^2\}$
$HsH = \{e, r, r^2\}\times\{s, sr, sr^2\} = \{s, sr, sr^2, rs, rsr, rsr^2, r^2s, r^2sr, r^2sr^2\}$
These are the only two double cosets (since they partition G).

(iii) $HeK = \{e, r, r^2\}\times\{e, s, rsr^2, r^2sr\}$
$= \{e, s, rsr^2, r^2sr, r, rs, r^2sr^2, sr, r^2, r^2s, sr^2, rsr\} = G$
Therefore there is only one double coset HgK.

Note: (for interest only)
$G \cong A_4$, (even permutations of S_4).
Map: $r \rightarrow (123)$
$s \rightarrow (12)(34)$

(b) (iii) You know that HeK is the whole of G because it has 12 elements (= $|G|$).

Question 16.

(a) $36 = 2^2 \times 3^2 = 4 \times 9$, $2400 = 2^5 \times 3 \times 5^2 = 32 \times 3 \times 25$

$A = Z_{36} \times Z_{2400} \cong (Z_4 \times Z_9) \times (Z_{32} \times Z_3 \times Z_{25}) \cong (Z_4 \times Z_{32}) \times (Z_3 \times Z_9) \times Z_{25}$

(i) 2-primary component is $Z_4 \times Z_{32}$
3-primary component is $Z_3 \times Z_9$
5-primary component is Z_{25}

(ii) $A \cong (Z_4 \times Z_3) \times (Z_{32} \times Z_9 \times Z_{25}) \cong Z_{12} \times Z_{7200}$ in canonical form

(b) $7200 = 2^5 \times 3^2 \times 5^2$

We seek subgroups of the p-primary components of A:

2-primary: order = 32, so Z_{32} or $Z_2 \times Z_{16}$ or $Z_4 \times Z_8$ (3 possibilities)
3-primary: order = 9, so Z_9 or $Z_3 \times Z_3$ (2 possibilities)
5-primary: order = 25, so Z_{25} (only 1 possibility)

Total of $3 \times 2 \times 1 = 6$ subgroups of A of order 7200.

(c) Largest order of a cyclic subgroup = highest possible order of any element $= \text{lcm}\{36, 2400\} = 7200$.

(d) $15 = 3 \times 5$, so calculate the number of order 3 elements in the 3-primary component × the number of order 5 elements in the 5-primary component.
Z_3 and Z_9 each have 2 elements of order 3.

If $(a, b) \in Z_3 \times Z_9$, and $|(a, b)| = 3$, then
$|a| = 1, |b| = 3$ (2 possibilities)
$|a| = 3, |b| = 1$ (2 possibilities)
$|a| = 3, |b| = 3$ (4 possibilities)
} 8 elements of order 3 in $Z_3 \times Z_9$

There are 4 elements of order 5 in Z_{25}
So there are $8 \times 4 = 32$ elements of order 15 in A.

(e) Only the last one represents A: $\begin{bmatrix} 32 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 75 \end{bmatrix}$

(a) (i) It doesn't matter whether you use Z_{2^2} or Z_4.

I use Z_4 here because it's easier typographically.

(a) (ii) NB: order matters for canonical form,
e.g. $Z_{7200} \times Z_{12}$ isn't canonical form

(c) You could have appealed to canonical form to state that the highest order is 7200.

(d) Elements of order p:
Z_p : $p-1$ elements of order p
Z_{p^2} : exactly one subgroup of order p ($\cong Z_p$), [Th[m] 4.2 (b), HB p25] so $p-1$ elements of order p.

(e) The 2-primary component is wrong in the other two.
First one: $Z_2 \times Z_{64}$,
Second one: $Z_2 \times Z_2 \times Z_{32}$

Question 17.

(a) $|G| = 275 = 5^2 \times 11$. Let n_p be the number of Sylow p-subgroups of G.
$n_{11} \equiv 1 \pmod{11}$ and $n_{11} \mid 25$, so $n_{11} = 1$
So the Sylow 11-subgroup, K, is unique, hence normal in G (Lemma 2.2), and has order 11. [$K \cong Z_{11}$ since 11 is prime.]

(b) $n_5 \equiv 1 \pmod 5$ and $n_5 \mid 11$, so $n_5 = 1$ or 11

(c) K is normal in G, so the quotient group G/K may be formed.
$|G/K| = 25$, so G/K is Abelian. (Result 5.2, HB p37). Thus $G/K \cong Z_5 \times Z_5$ or $G/K \cong Z_{25}$ and in either case has a normal subgroup N, of order 5.
By the Correspondence Theorem, N gives rise to a corresponding normal subgroup of G containing K, with order $5|K| = 55$.

(d) Let H be the normal subgroup of G of order 25. Then $|K| \times |H| = |G|$, and 11 and 25 are coprime. Therefore $G \cong H \times K$, by Th[m] 4.1, HB p49.

We have $K \cong Z_{11}$ and $H \cong Z_5 \times Z_5$ or $H \cong Z_{25}$

So $G \cong Z_{11} \times Z_5 \times Z_5 \cong Z_5 \times Z_{55}$ or $G \cong Z_{11} \times Z_{25} \cong Z_{275}$.

(a) & (b) Theorem 3.1 (Summary of the Sylow results) HB p48

(c) ***Note*:** If N is a subgroup of G/K, then $N = N'/K$, where N' is the corresponding subgroup of G containing K i.e. N is the set of cosets of K in N'.

If $|N| = 5$, there are 5 cosets of K in N'. Since there are 11 elements of G in each coset, and the cosets partition N', there must be 55 elements of G in N'

[P&S section 3.9, p 23 – 25]