M336 2004 Exam Solutions

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Health Warning: This is NOT official OU material. The solutions show how I personally would have tackled the paper. They have not been verified, and it is unlikely that they would agree totally with the Exam Board's answers. (Let me know if you spot any errors)

Part I (55 marks - 90 minutes)

Question 1.

(a) Tile types: [4, 5, 5], [5, 5, 5, 5]
Vertex types: (3, 3, 3, 3), (3, 3, 4, 4, 4)

(b) Not edge-to-edge, not monohedral

(c) [5, 5, 5, 5]

(b) Tile type [5, 5, 5, 5] has 'bent' edges, so not edge-to-edge. (Definition: HB p8)

(c) Stars and hexagons are the same type, [5, 5, 5, 5], but shapes, and therefore orbits, differ.

Question 2.

(a) $sr^3s^{-1} = (sr^3)s^{-1} = (r^{-3}s)s^{-1} = r^{-3} = r^5$ (since $r^8 = e$)

(b) $sr^m = r^m s \Rightarrow sr^m s^{-1} = r^m$

From (a), we can deduce that $sr^m s^{-1} = r^{-m}$, so we need to find m, such that $r^m = r^{-m}$. Since $r^8 = e$, only e and r^4 commute with s.

(a) I used *Result* 3.3.2 (HB Supp). You could have done it the conscientious way:

$(sr)r^2s^{-1} = (r^{-1}s)r^2s^{-1} =$

$r^{-1}(sr)rs^{-1} = r^{-1}(r^{-1}s)rs^{-1}$

etc

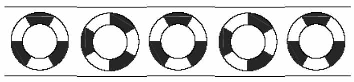
Question 3.

(a) v ✓ h ✗ g ✓ , so Type 7

(b) v ✓ h ✓ , so Type 6

(c) Type 1, since all symmetries, except the translations, are lost.

Algorithm: HB p14



Question 4.

Since H has index 2 in G, there are exactly 2 left cosets, $\{H, aH\}$, and exactly 2 right cosets, $\{H, Ha\}$, of H for all $a \in G$.

If $a \in H$, then $aH = H = Ha$, (since H is closed).

If $a \notin H$, the elements of the coset aH consists of all elements of G which are not in H, since H and aH are disjoint (the two cosets partition H).

But these are precisely the elements of Ha, by the same argument. So we must have $aH = Ha$ for all $a \in G$, and H is normal in G.

Note: this result is in the Handbook Supplement.

HB p17 *Lemma* 2.1

HB p17 *Theorem* 2.7

Question 5.

(a) $G \cong D_3$, so $|G| = 6$.

g	example cycle	cs(g)	number
e	(1)(2)(3)(4)(5)(6)	x_1^6	1
r, r^2	(123)(456)	x_3^2	2
s, rs, r^2s	(3)(4)(12)(56)	$x_1^2x_2^2$	3

Cycle index: $\frac{1}{6}\left(x_1^6 + 2x_3^2 + 3x_1^2x_2^2\right)$

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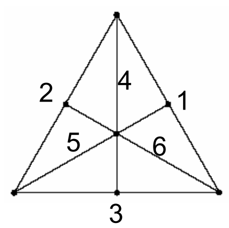
Label the objects:



NB: there are 6 objects in the set that G is acting upon, so if cs(g) = $x_a^b x_c^d$, then $(a \times b) + (c \times d) = 6$

HB Supp p2.

Question 5 (continued).

(b)

g	example cycle	cs(g)	number
e	(1)(2)(3)(4)(5)(6)(7)	x_1^7	1
r, r^2	(7)(123)(456)	$x_1x_3^2$	2
s, rs, r^2s	(1)(5)(7)(46)(23)	$x_1^3x_2^2$	3

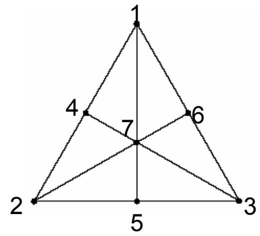
Cycle index: $\frac{1}{6}\left(x_1^7 + 2x_1x_3^2 + 3x_1^3x_2^2\right)$

(c)

g	example cycle	cs(g)	number
e	(1)(2)(3)(4)(5)(6)	x_1^6	1
r, r^2	(123)(456)	x_3^2	2
s, rs, r^2s	(14)(26)(35)	x_2^3	3

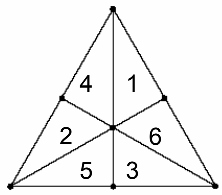
Cycle index: $\frac{1}{6}\left(x_1^6 + 2x_3^2 + 3x_2^3\right)$

(b)



NB: 7 objects in this one.

(c)



Question 6.

(a) 175 = 165 + 10
165 = 16×10 + 5
10 = 2×5 + 0

hcf{175, 165} = 5

(b) 5 = 165 – (16×10)
= 165 – 16(175 – 165)
= 17×165 - 16×175

(c) $\text{lcm}\{175, 165\} = \dfrac{175 \times 165}{\text{hcf}\{175, 165\}} = \dfrac{175 \times 165}{5} = 5775$

Algorithm HB p22

NB: the method in Frame 10 (GR1 p21) uses the correct algorithm. ***Do not*** follow the method used in Frame 6A (p19).

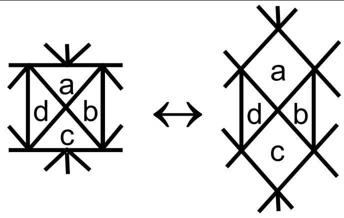
Theorem 2.2, HB p22

Question 7.

(a) $\mathfrak{I}_B$ and $\mathfrak{I}_C$ are isomorphic

(b) $n_v(\mathfrak{I}_A) = 2$, $n_t(\mathfrak{I}_A) = 4$

(c) By Euler's equation:
$n_e(\mathfrak{I}_A) = n_t(\mathfrak{I}_A) + n_v(\mathfrak{I}_A) = 4 + 2 = 6$



Tile types [4,5,5] and [4,5,5,5] are arranged in the same way

Euler: *Theorem* 2.2, HB p29

Question 8.

(a)
(b)

$$\begin{bmatrix} 2 & -2 & -4 \\ 6 & 6 & 0 \\ 4 & 8 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 & -4 \\ 0 & 12 & 12 \\ 0 & 12 & 12 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 12 & 12 \\ 0 & 12 & 12 \end{bmatrix} \quad \begin{matrix} C_2 \rightarrow C_2 + C_1 \\ C_3 \rightarrow C_3 + 2C_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 12 & 12 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad C_3 \rightarrow C_3 - C_2$$

(c) $A \cong Z_2 \times Z_{12} \times Z$, so torsion coefficients are 2 and 12, order is infinite.

Theorem 3.2 HB p33

NB: It's a good idea to show exactly what you were trying to do, so that you can get some of the marks even if you commit arithmetic hari kiri doing the actual calculations.

Warning: Easy to forget that the zeros in the bottom row give you the extra Z term, so A has infinite order

$Z_2 \times Z_{12} \times Z$ is already in canonical form. If it weren't, you'd have to put it into canonical form to find the torsion coefficients.

Question 9.

(a) $\mathbf{a} = \mathbf{a} + 0\mathbf{b}$, $\mathbf{c} = 2\mathbf{a} + \mathbf{b}$. The transition matrix from $\{\mathbf{a,c}\}$ to $\{\mathbf{a,b}\}$ is

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ with integer entries and determinant equal to 1. So $L(\mathbf{a,c}) = L(\mathbf{a, b})$.

(b) $\mathbf{b} = 0\mathbf{a} + \mathbf{b}$, $\mathbf{c} = 2\mathbf{a} + \mathbf{b}$. The transition matrix from $\{\mathbf{b,c}\}$ to $\{\mathbf{a,b}\}$ is

$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ with determinant equal to −2. So $L(\mathbf{b,c}) \neq L(\mathbf{a, b})$.

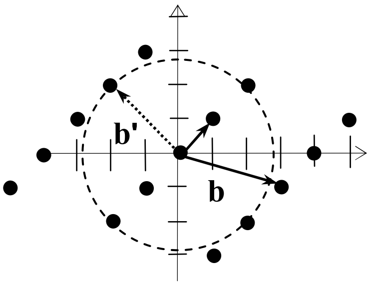
(c) Reduced basis: $\{\mathbf{a}, \mathbf{b'}\} = \{(1, 1), (-2, 2)\}$ [*NB: smallest first*]

$\|\mathbf{a}\| = \sqrt{2}$, $\|\mathbf{b'}\| = \sqrt{8}$, $\mathbf{a} \cdot \mathbf{b'} = 0$, so L is rectangular.

Equality of lattices: *Theorem* 1.2, HB p38

(c) ***NB: Draw a picture***, with lots of lattice points.

Classification: HB Supp p3



Question 10.

$|G| = 39$, and $Z(G)$ is a normal subgroup of G, so $|Z(G)|$ divides 39. Hence $|Z(G)| \in \{1, 3, 13, 39\}$.

$|Z(G)| < |G|$, since G is non–Abelian. So $|Z(G)| \neq 39$.

If $|Z(G)| \in \{3, 13\}$, then $|G / Z(G)| \in \{13, 3\}$, and the quotient group would be cyclic (prime order). This implies that G is Abelian – a contradiction.

Therefore $|Z(G)| = 1$, and the centre is trivial.

Result 4.2, HB p37

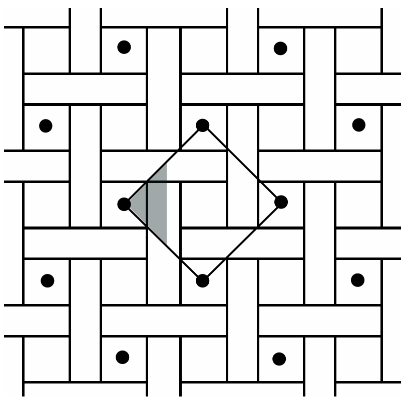
Theorem 4.3 HB p37

Question 11.

(a) Highest order of rotation is 4

(b) W has reflections, so type is $p4mm$ or $p4mg$. In either case, the point group is D_4.

(c)



HB p45 for classification
HB p46 for point groups

I've added the points of the associated lattice, to show you how I contrived to get a basic *square*.

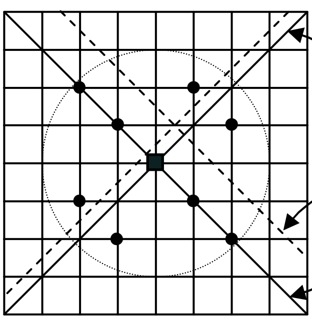
$|D_4| = 8$, so the area of the generating region is $1/8 \times$ the area of the basic square.

HB Supplement, pp 3 – 4.

Part II A (Geometry)

Question 12

(a) & (c)



Note: The axis of $q[\pi/4]$ passes through (¾, ¾), and the lattice point $\mathbf{a} + \mathbf{b} = (3, 3)$ lies on this axis.

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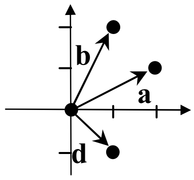
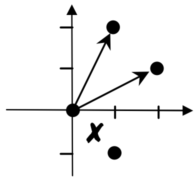
Question 12 (continued)

(b) Reduced basis: $\{\mathbf{d}, \mathbf{a}\} = \{(1, -1), (2, 1)\}$
$\|\mathbf{d}\| = \sqrt{2}$, $\|\mathbf{a}\| = \sqrt{5}$, $\mathbf{d} \cdot \mathbf{a} = 1 = \tfrac{1}{2}\|\mathbf{d}\|^2$
So L is rhombic.



(d) (i) $r' = r[(\tfrac{1}{2}, -\tfrac{1}{2}), \pi] = t[(1, -1)]r[\pi]$
(since the origin maps to $(1, -1)$)
$t[(1,-1)] = t[\mathbf{a} - \mathbf{b}] = t[\mathbf{a}]t[-\mathbf{b}]$, so $r' = t_a t_b^{-1} r$



(ii) *First choice*:
Reflection component is $q[(\tfrac{3}{4}, \tfrac{3}{4}), \pi/4] = q[\pi/4] = q$ (fixes (0,0))
Translation component maps (0,0) to any lattice point on the line $y = x$ (i.e. in same direction as axis), say **a + b,** giving $t[\mathbf{a} + \mathbf{b}]$. So $q' = t_a t_b q$.

Second choice:
Reflection component is $q[(\tfrac{3}{4}, \tfrac{3}{4}), 3\pi/4]$, maps (0,0) to $(1\tfrac{1}{2}, 1\tfrac{1}{2})$
Translation component maps $(1\tfrac{1}{2}, 1\tfrac{1}{2})$ to a lattice point in the same direction as the glide axis, say $\mathbf{b} = (1, 2)$ (so translation component would be $t[(-\tfrac{1}{2}, \tfrac{1}{2})]$)
So image of (0, 0) under q' is **b**, and standard form is $q' = t[\mathbf{b}]q[3\pi/4]$
$q[3\pi/4] = r[\pi]q[\pi/4] = rq$, so $q' = t_b rq$.

(e) *First choice:*
Reflection component: $q[\pi/4]$, Translation component: $t[\boldsymbol{a} + \mathbf{b}]$
Both are symmetries of the lattice, so q' is an inessential glide.
Second choice:
Reflection component: $q[(\tfrac{3}{4}, \tfrac{3}{4}), 3\pi/4]$,
Translation component: $t[(-\tfrac{1}{2}, \tfrac{1}{2})]) = t[-\tfrac{1}{2}(\mathbf{a} - \mathbf{b})]$
Neither is a symmetry of the lattice, so q' is an essential glide.

(b) HB Supp. p3

(d) (i) Standard form the quick way: *P&S*, p14

(d) (ii) ***Ambiguous question***! It doesn't tell you where the origin should go.

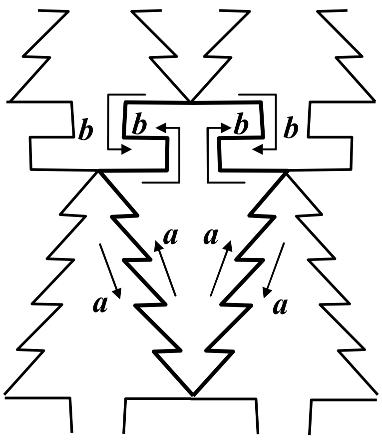
So two choices for reflection component, (at angles $\pi/4$ or $3\pi/4$) and infinitely many choices for translation component (to any lattice point in the same direction as the reflection axis). See diagram in part (a).

Full marks for whichever route you took – provided your answer was consistent and correct.

[***Note***: *for the second choice, you could have picked* **a** *instead of* **b**, *giving translation component* $t[(\tfrac{1}{2}, -\tfrac{1}{2})]$, *and* $q' = t_a rq$]

Question 13.

(a) (i)

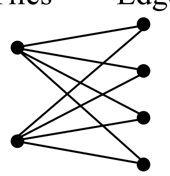


Note: the edges labelled ***b*** consist of 5 segments, but each is a single edge.

(ii) $a^{\rightarrow}$ $b^{\rightarrow}$ $b^{\leftarrow}$ $a^{\leftarrow}$
$a^{\leftarrow}$ $b^{\leftarrow}$ $b^{\rightarrow}$ $a^{\rightarrow}$

Incidence symbol: HB p31

(iii)



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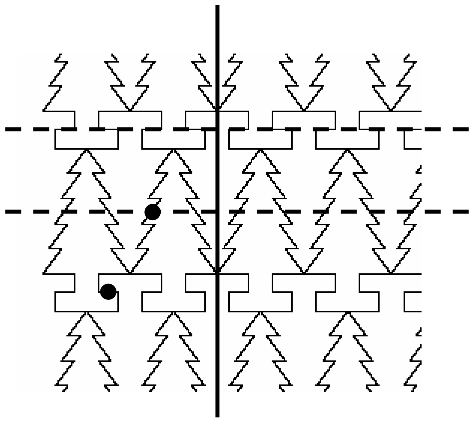
2 *translational* tile orbits (the two orientations of the tree)

4 *translational* edge orbits (i.e. the inside edges labelled **a,b,b,a** in the diagram above.)

Question 13 (continued)

(a) (iv) Two translational vertex orbits (at the sharp end of each orientation of the tree), give a one-to-one correspondence between vertex orbits and tile orbits. Each tile and vertex is incident with 4 edges, one from each of the 4 edge orbits. So the diagrams are isomorphic.

(b) (i)



(ii) Highest order? 2
Reflections? Yes
Two directions? No Therefore *p*2*mg*

Note that the question asks for orbits under $\Gamma(\wp)$, i.e. the full symmetry group, not just the translations.

So, for instance, reflection axes through upside down trees rotate to axes through upright trees.

Algorithm: HB p45

Question 14.

(a) (i) $\mathbf{c} = (0, 0, 2) + \tfrac{3}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}$, where (0, 0, 2) is orthogonal to **a** and **b.**
So offset = (½, ½).

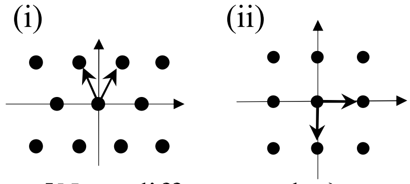
$\|\mathbf{a}\| = \|\mathbf{b}\| = \sqrt{5}$, $\mathbf{a}\cdot\mathbf{b} = 3 \neq \tfrac{1}{2}\|\mathbf{a}\|^2$, so $L(\mathbf{a}, \mathbf{b})$ is rhombic, and $\{\mathbf{a}, \mathbf{b}\}$ is a suitable basis. $L(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is *face-centred orthorhombic*. (Result 3.2)

(ii) $\mathbf{c} = (0, 0, 2) + \tfrac{1}{2}\mathbf{a} - \mathbf{b}$, where (0, 0, 2) is orthogonal to **a** and **b.**
So offset = (½, 0).

$\|\mathbf{a}\| = \|\mathbf{b}\| = 1$, $\mathbf{a}\cdot\mathbf{b} = 0$, so $L(\mathbf{a}, \mathbf{b})$ is square, $\{\mathbf{a}, \mathbf{b}\}$ is reduced, hence a suitable basis, so $L(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is *base-centred orthorhombic*.

2-d lattices: HB Supp p3
Bravais: HB Supp p4
Result 3.2, HB p56.

(i) (ii)



[*Note*: different scales}

(b) (i) Labelling short edges a, b, c, d, e, f and long edges 1, 2, 3, 4, 5, 6:

rotation	**example cycle**	**cs(g)**	**number**
e	(a)…(e)(1)…(6)	x_1^{12}	1
π (*opposite edges*)	(ad)(be)(cf)(1)(4)(26)(35)	$x_1^2 x_2^5$	3
2π/3 (*opposite faces*)	(abc)(def)(123)(456)	x_3^4	2

Cycle index: $\frac{1}{6}\left(x_1^{12} + 3x_1^2x_2^5 + 2x_3^4\right)$,

With *m* colours: $\frac{1}{6}\left(m^{12} + 3m^7 + 2m^4\right)$

(ii) Short black rods can go only into the top or bottom face .

There is 1 prism with no black rods, 1 with one black rod, and 1 with two black rods in the same face (top or bottom).

There are 3 prisms with one black rod in the top and one in the bottom (fix the one at the top, say, then there are 3 places where the one at the bottom can go.)

Total: 1 + 1 + 1 + 3 = 6.

(iii) $G^+ \simeq D_3$, and central inversion is a symmetry of the antiprism.
So $G \simeq D_3 \times C_2$

(b) (i) Short edges only in top and bottom faces, so each must map to itself (rotation through ±2π/3, axis through centres of these faces) or to the opposite face (rotation through π, axis through opposite upright edges).

(b) (ii) If you had a couple of days to spare, you could have worked out the cycle index for the short rods only, then worked out the coefficients of W^6, BW^5 and B^2W^4 from the pattern inventory.

(b) (iii) G^+ has 6 elements; 2 of order 3, 3 of order 2, so $G^+ \simeq D_3$

Classification: HB p53

Part II B (Groups)

Question 15

(a) For each $g \in G$ and $x \in G$,

- $g \wedge x = gxg^{-1} \in G$, since G is closed
- $e \wedge x = exe^{-1} = x$
- $(gh) \wedge x = (gh)x(gh)^{-1} = (gh)x(h^{-1}g^{-1}) = g(hxh^{-1})g^{-1} = g \wedge h \wedge x$

Hence this defines a group action.

Definition of a group action: Lemma 5.1, HB p10

The orbit of the element x is the conjugacy class to which x belongs. By the Orbit-Stabilizer theorem, the number of elements in the orbit (hence the number of elements in the conjugacy class) divides the order of G.

Orbit-Stabilizer Theorem: Theorem 3.2, HB p27.

(b) Suppose $|G| = n$, and there are two conjugacy classes. One class contains the single element, e, so the other class contains $n - 1$ elements (by the class equation).

By part (a) above, $n - 1 \mid n$, so $n = 2$, and $|G| = 2$. Hence $G \cong Z_2$.

Class equation: Theorem 4.1, HB p36

(c) If G is Abelian, then each conjugacy class contains a single element, and the class equation with 3 conjugacy classes would give $|G| = 3$.
So G is not Abelian.

The class equation for G is: $2s = 1 + m + n$, so $m + n = 2s - 1$.

$m \neq n$ (since $m + n$ is odd), so we may assume $m < n$.

We have $n \mid 2s$, i.e. $n \mid 1 + m + n$, so $n \mid m + 1$, and therefore $n = m + 1$.

Similarly, $m \mid 1 + m + n$, so $m \mid 1 + n$, i.e. $m \mid m + 2$, so $m = 1$ or $m = 2$.

If $m = 1$, then $n = 2$ and $2s = 1 + 1 + 2 = 4$, so $s = 2$.
Hence $|G| = 4$, and $G \cong Z_4$ or $G \cong Z_2 \times Z_2$, both of which are Abelian.

Therefore, we must have $m = 2$, $n = 3$, $2s = 1 + 2 + 3 = 6$, so $s = 3$.
Therefore $|G| = 6$, and $G \cong S_3$ ($\cong D_3$).

S_3 is the only non-Abelian group of order 6. HB p26

Question 16.

(a) $|G| = 500 = 2^2 \times 5^3$

prime	factors	label
2	2^2	2a
	2×2	2b
5	5^3	5a
	5×5^2	5b
	$5 \times 5 \times 5$	5c

There are $2 \times 3 = 6$ possible groups.

label	p-primary form	canonical form
1) 2a5a:	$Z_4 \times Z_{125}$	Z_{500}
2) 2b5a:	$Z_2 \times Z_2 \times Z_{125}$	$Z_2 \times Z_{250}$
3) 2a5b:	$Z_4 \times Z_5 \times Z_{25}$	$Z_5 \times Z_{100}$
4) 2b5b:	$Z_2 \times Z_2 \times Z_5 \times Z_{25}$	$Z_{10} \times Z_{50}$
5) 2a5c:	$Z_4 \times Z_5 \times Z_5 \times Z_5$	$Z_5 \times Z_5 \times Z_{20}$
6) 2b5c:	$Z_2 \times Z_2 \times Z_5 \times Z_5 \times Z_5$	$Z_5 \times Z_{10} \times Z_{10}$

It doesn't matter whether you use Z_{2^2} or Z_4. I use Z_4 here because it's easier to type.

***NB*: order matters for canonical form, e.g. $Z_{250} \times Z_2$ isn't canonical form**

(b) $50 = 2 \times 5^2$, so $Z_{50} \cong Z_2 \times Z_{25}$

All the groups have a subgroup isomorphic to Z_2, but only 1), 2), 3) and 4) have a subgroup isomorphic to Z_{25}. So all the groups except 5) and 6) have a subgroup isomorphic to Z_{50}.

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Question 16 (continued).

(c) (i) Z_{25} and Z_{125} each have 20 elements of order 25.

(ii) We may write $G \cong A \times B$, where A is the 2-primary component, B is the 5-primary component, and $|(a, b)| = \text{lcm}\{|a|, |b|\}$, $a \in A$, $b \in B$.

Since $50 = 2 \times 25$, we must have $|a| = 2$, $|b| = 25$, so we calculate the number of elements of order 2 in the 2-primary component × the number of elements of order 25 in the 5-primary component.

2-primary component	no. of order 2 elements	5-primary component	no. of order 25 elements
Z_4	1	Z_{125}	20
$Z_2 \times Z_2$	3	$Z_5 \times Z_{25}$	$5 \times 20 = 100$

1) $Z_4 \times Z_{125}$ — $1 \times 20 = 20$
2) $(Z_2 \times Z_2) \times Z_{125}$ — $3 \times 20 = 60$
3) $Z_4 \times (Z_5 \times Z_{25})$ — $1 \times 100 = 100$
4) $(Z_2 \times Z_2) \times (Z_5 \times Z_{25})$ — $3 \times 100 = 300$

Groups 5) and 6) have no subgroup (and therefore no element) of order 25, hence no element of order 50.

Z_5 : 4 elements of order 5

Z_{25}: exactly one subgroup of order 5 [*Th*^m 4.2 (b), HB p25] so 4 elements of order 5, and the other 20 non-identity elements are of order 25.

In $Z_5 \times Z_{25}$, each of the 20 elements of order 25 in Z_{25} may be combined with each of the 5 elements in Z_5, so $5 \times 20 = 100$ elements of order 25.

Z_{125}: exactly one subgroup of order 25, so 20 elements of order 25.

Z_2 and Z_4 each have 1 element of order 2,

$Z_2 \times Z_2$ has no element of order 4, so the 3 non-identity elements have order 2.

Question 17.

(a) $|G| = 175 = 5^2 \times 7$.

Let n_p be the number of Sylow p-subgroups of G. Then

$n_5 \equiv 1 \pmod 5$ and $n_5 \mid 7$, so $n_5 = 1$

$n_7 \equiv 1 \pmod 7$ and $n_7 \mid 25$, so $n_7 = 1$

Summary of the Sylow results: *Th*^m 3.1 HB p48

(b) For any group G of order 175:

- the Sylow 5-subgroup, H, and the Sylow 7-subgroup, K, are both unique, hence normal in G
- $|H| = 25$ and $|K| = 7$, 25 and 7 are coprime
- $|G| = |H| \times |K|$ $(175 = 25 \times 7)$

So $G \cong H \times K$, by *Theorem* 4.1, HB p49.

H is Abelian (order is the square of a prime), so $H \cong Z_{25}$ or $H \cong Z_5 \times Z_5$

K is cyclic (prime order), so $K \cong Z_7$.

Hence $G \cong Z_{25} \times Z_7 \cong Z_{175}$, or $G \cong Z_5 \times Z_5 \times Z_7 \cong Z_5 \times Z_{35}$.

In either case, G is the direct product of Abelian groups, and is therefore Abelian.

Lemma 2.2 HB p48

Note: you could have quoted *Result* 1.1, HB p49, to establish that $G \cong H \times K$

Result 5.2 HB p37

Theorem 4.1, HB p22