M336 1994 Exam Solutions

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<u>Health Warning</u>: This is NOT official OU material. They have not been verified, and it is unlikely that they would agree totally with the Exam Board's answers. (Let me know if you spot any errors)

Part I Questions (55 marks - 90 minutes)

Question 1.

(a) Tile types: [3, 3, 3, 3] [3, 3, 3, 3, 3, 4]
Vertex types: (4, 6, 6) (6, 6, 6) (6, 6, 6, 6)

(b) i) Not vertex uniform (vertices of degrees 3 and 4)
ii) $\mathfrak{I}'$ **is** tile uniform

HB 9-10
Don't bother to memorise which brackets to use.

Question 2.

$(a)\quad (r^2s)(rs) = r^2(sr)s = r^2(r^3s)s = r^5s^2 = rr^2$ (since $r^4 = e,\ s^2 = r^2$)
$= r^3$

$(b)\quad (r^3s)^{-1} = (sr)^{-1}$ (since $sr = r^3s$)
$= r^{-1}s^{-1} = r^3s^3$ (since $r^4 = s^4 = e$)
$= r^5s$ (since $s^2 = r^2$)
$= rs$ (since $r^4 = e$)

Watchpoint:
Remember that $(xy)^{-1} = y^{-1}x^{-1}$

Question 3.

v? no; h? no; g? yes; So Type 4

HB 14 Algorithm

Question 4.

Closure: Let $(h_1, k_1), (h_2, k_2) \in H \times K$. Then
$(h_1, k_1)(h_2, k_2) = (h_1h_2, k_1k_2) \in H \times K$ (since H, K are close)
Identity: Identity of $G \times G$ is (e, e)
$e \in H$ and $e \in K$ (subgroups), so $(e, e) \in \mathrm{H} \times K$.
Inverses: $(h, k)^{-1} = (h^{-1}, k^{-1})$ in $G \times G$.
$h^{-1} \in H$, $k^{-1} \in K$ (subgroups), so $(h^{-1}, k^{-1}) \in H \times \mathrm{K}$
Hence $H \times \mathrm{K}$ is a subgroup of $G \times G$.

HB 18 Direct product

Question 5.

Direct symmetries of V: $\{e, r\}$
Orbits: {1, 7}, {2, 8}, {3, 9}, {4, 10}, {5, 11}, {6, 12}

All symmetries of V: $\{e, r, v, h\}$
Orbits: {1, 5, 7, 11}, {2, 4, 8, 10}, {3, 9}, {6, 12}

All symmetries of D_4: $\langle r, s : r^4 = s^2 = e, sr = r^3s \rangle$
Orbits: {1, 2, 4, 5, 7, 8, 10, 11}, {3, 6, 9, 12}

Question 6.

$G = Z_{10} \times Z_{15}$ is Abelian (product of cyclic groups), so all subgroups are Abelian. Z_6 is the only Abelian group of order 6, with 2 elements of order 6. G has only 2 elements of order 6, (5, 5) and (5, 10), ie enough for only 1 subgroup of order 6.

Note: $|5| = 2$ in Z_{10}, and $|5| = |10| = 3$ in Z_{15}. So $|(5, 5)| = |(5, 10)| = 6 = \mathrm{lcm}(2, 3)$ in G. [Result 2.6, HB 22.]

Question 7.

(a) $n_t(\mathfrak{I}) = 6$

(b) $\mathfrak{I}$ is periodic and tile-uniform, with tile type [3, 4, 6, 4], so
$n_v(\mathfrak{I}) = 6(1/3 + 1/4 + 1/6 + 1/4) = 6$ **Continued on next page**

Tiles in each hexagon have different orientations
HB 29 Theorem 3.2

Question 7 (continued).

(c) Tiles have 4 edges, so 4 lines from each tile orbit in the diagram, giving a total of $6 \times 4 = 24$ lines in the diagram.
Each edge has 2 tiles incident with it, so 2 lines enter each edge orbit in the diagram, ie there must be $24/2 = 12$ edge orbits.

Check:. HB 29, Th'm 2.2: (Euler)
$n_t(\mathfrak{I})+n_v(\mathfrak{I}) = n_e(\mathfrak{I})$
$6 + 6 = 12$ ✓

Question 8.

(a) $A = Z_6 \times Z_8 \times Z_9 = (Z_2 \times Z_3) \times Z_8 \times Z_9 = Z_6 \times Z_{72}$

(b) 2-primary component: $Z_2 \times Z_8$; 3-primary component: $Z_3 \times Z_9$

HB 33 Theorem 3.2

HB 38 Primary comp.

Question 9.

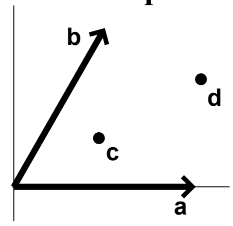
(a) $q[\boldsymbol{a}, 2\pi/3]$

(b) $r[½(\boldsymbol{a}+\boldsymbol{b}), \pi]$

(c) $r[\boldsymbol{b}, \pi/3]$ (or $r[\boldsymbol{a}, -\pi/3]$)

(d) $q[½\boldsymbol{a}, ½\boldsymbol{b}, 0]$ (or $t[½\boldsymbol{a}]q[½\boldsymbol{b}, 0]$)

NB: Draw a picture



Question 10.

prime	factors	label
3	3^2	3a
	3×3	3b
2	2^2	2a
	2×2	2b

2a3a: $Z_4 \times Z_9 \cong Z_{36}$

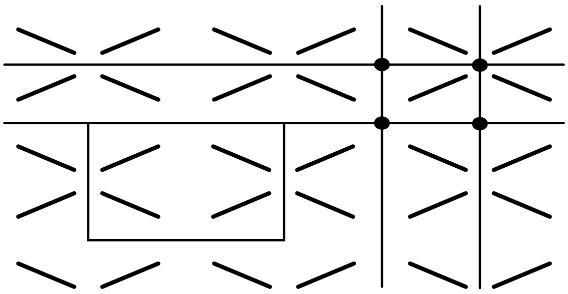
2b3a: $Z_2 \times Z_2 \times Z_9 \cong Z_2 \times Z_{18}$

2a3b: $Z_4 \times Z_3 \times Z_3 \cong Z_3 \times Z_{12}$

2b3b: $Z_2 \times Z_2 \times Z_3 \times Z_3 \cong Z_6 \times Z_6$

HB 36 Result 1.1

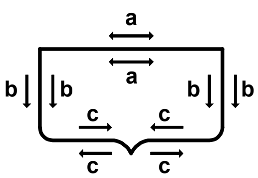
Question 11.



HB 47 Associated lattice, Basic parallelogram etc

Part IIA (Geometry)

Question 12.



(b) $\boldsymbol{a}\ \ \boldsymbol{b}^{\rightarrow}\ \ \boldsymbol{c}^{\rightarrow}\ \ \boldsymbol{c}^{\leftarrow}\ \ \boldsymbol{b}^{\leftarrow}$
$\boldsymbol{a}\ \ \boldsymbol{b}^{\rightarrow}\ \ \boldsymbol{c}^{\leftarrow}\ \ \boldsymbol{c}^{\rightarrow}\ \ \boldsymbol{b}^{\leftarrow}$

(c) 2 appearances of each letter, so order = 2
Not all right arrows, so group is dihedral
Tile stabiliser must be D_1.

(c) $\boldsymbol{a}$: same and undirected, so stabiliser is { *e, r, v, h* }
$\boldsymbol{b}$: same and similarly directed, so stabiliser is { *e, h* }
$\boldsymbol{c}$: same but oppositely directed, so stabiliser is { *e, r* }

Note: The given arrow for edge a is double-headed, so expect other symbols to appear twice (or more double-headed arrows)

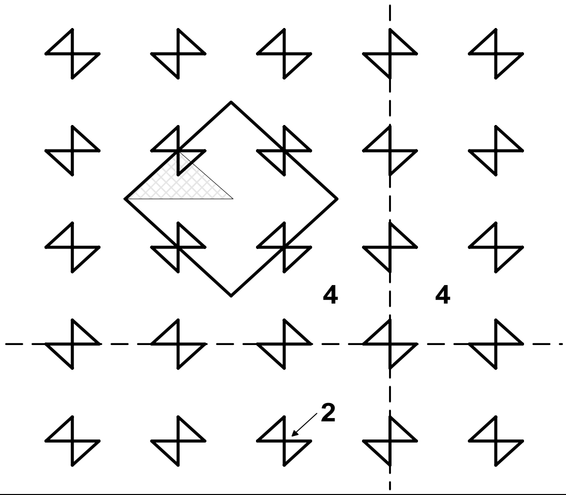
HB 31 Incidence symbol

HB 30 Theorem 5.1
Note: D_1 is *trivially* cyclic.
HB 30 Theorem 5.2

Question 13.

(a) Highest order of rotation? 4 (associated lattice is square)
Any reflections? yes
Reflections in 4 directions? yes (same symmetries as square lattice)
Therefore type is p4mm

(b)



HB 45 Algorithm

Watchpoint: Orientation of the associated lattice.
To draw the basic parallelogram, pick an origin, mark all the points the origin can be **translated** to before deciding how to draw the basic parallelogram.

Question 14.

(a) $L(\boldsymbol{b}, \boldsymbol{c})$ is a square lattice (in the *yz* plane) $\|\boldsymbol{b}\| = \|\boldsymbol{c}\| = 1$, $\boldsymbol{b} \cdot \boldsymbol{c} = 0$
$\boldsymbol{a} = (2, 0, 0)$, so offset relative to $(\boldsymbol{b}, \boldsymbol{c})$ is $(0, 0)$, $\|\boldsymbol{a}\| = 2 > \|\boldsymbol{b}\|$
Therefore the lattice is **primitive tetragonal.**

(b) $L(\boldsymbol{a}, \boldsymbol{b})$ is rectangular, since $\|\boldsymbol{a}\| = 1$, $\|\boldsymbol{b}\| = 3$ and $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
$\boldsymbol{c} = (0, 0, 2) + \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$, so offset relative to $(\boldsymbol{a}, \boldsymbol{b})$ is $(\frac{1}{2}, \frac{1}{2})$
Therefore the lattice is **body-centred orthorhombic**

(c) $L(\boldsymbol{a}, \boldsymbol{b})$ is rhombic, since $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \sqrt{5}$, $\boldsymbol{a} \cdot \boldsymbol{b} = 3 \neq {}^{1}/_{2}\|\boldsymbol{a}\|^2$.
$\mathbf{c} = (0, 0, 3) + \frac{1}{2}\boldsymbol{a} + (-1+\frac{1}{2})\boldsymbol{b}$, so offset $= (\frac{1}{2}, \frac{1}{2})$
Therefore the lattice is **face-centred orthorhombic**

(d) $L(\boldsymbol{a}, \boldsymbol{b})$ is hexagonal, since $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = 2$, $\boldsymbol{a} \cdot \boldsymbol{b} = 2 = \frac{1}{2}\|\boldsymbol{a}\|^2$.
$\mathbf{c} = (0, 0, {}^{3}/_{2}) + \vartheta\boldsymbol{a} + \vartheta\boldsymbol{b}$, so offset $= (\vartheta, \vartheta)$
Vertical distance $= {}^{3}/_{2} = {}^{3}/_{4}\|\boldsymbol{a}\|$, so the lattice is **trigonal.**

HB 56 Result 3.2
HB 40 Theorem 5.3

Watchpoint:
Don't just use the table blindly - you need to take account of **orientation,** so the 2-dim lattice may not always be in the *xy* plane, eg part (a).

Part IIB (Groups)

Question 15.

(a) i) Let $a \in H \cap Z(G)$, then $a \in H$ and $a \in Z(G)$.
For all $h \in H$, $ah = ha$, since $h \in G$ and $a \in Z(G)$, so $a \in Z(H)$
ie $H \cap Z(G) \subseteq Z(H)$

ii) Let $G = D_6$, $Z(G) = \{e, r^3\}$, $H = \{e, r^2, r^4\}$, $Z(H) = H$
$H \cap Z(G) = \{e\}$, so $Z(H) \subseteq H \cap Z(G)$ is **false**.

(b) i) **False**
eg let $G = Z_4$, so $Z(G) = G$ (Abelian), $H = \{0, 2\}$ and $Z(H) = H$.

ii) **True**
Let $h \in Z(H_1) \cap Z(H_2)$, then $h \in Z(H_1)$ and H_1, $h \in Z(H_2)$ and H_2, so $h \in H_1 \cap H_2$. Thus h commutes with every element of H_1, and therefore with every element of $H_1 \cap H_2$, so $h \in Z(H_1 \cap H_2)$.
ie $Z(H_1) \cap Z(H_2) \subseteq Z(H_1 \cap H_2)$.

HB 37 Centre of a group (ie the set of elements that commute with **every** element in the group)

Note: in (a) ii), *H* is Abelian (but D_6 is not).

For (b) i) you could use any Abelian group, as long as it has a non-trivial subgroup.

Solution	Reference
Question 16.	
(a) $\|G\| = 700 = 2^2 \times 5^2 \times 7$	HB 33 canonical form HB 36 p-primary form

prime	factors	label
7	7	7a
5	5^2	5a
	5×5	5b
2	2^2	2a
	2×2	2b

label	*p*-primary form	canonical form
2a5a7a:	$Z_4 \times Z_{25} \times Z_7$	Z_{700}
2b5a7a:	$Z_2 \times Z_2 \times Z_{25} \times Z_7$	$Z_2 \times Z_{350}$
2a5b7a:	$Z_4 \times Z_5 \times Z_5 \times Z_7$	$Z_5 \times Z_{140}$
2b5b7a:	$Z_2 \times Z_2 \times Z_5 \times Z_5 \times Z_7$	$Z_{10} \times Z_{70}$

(b) Each of the groups in part (a) may be expressed in the form $H \times Z_7$, where $|H| = 100$ (see the *p*-primary forms above, where H is the subgroup containing the 2-primary and 5-primary components of each of the groups).

No element in a group of order 100 can have an element of order 7 (by Lagrange's theorem), so each group can have only 1 subgroup of order 100, isomorphic to $H \times \{e\}$ in each case.

Question 17.

(a) $|G| = 30 = 2 \times 3 \times 5$. Let n_p be the number of Sylow *p*-subgroups of G. *(HB 48 Theorem 3.1)*

Then by Sylow's theorems:

$n_3 \equiv 1 \pmod 3$ and $n_3 \mid 10$, so $n_3 = 1$ or 10.

$n_5 \equiv 1 \pmod 5$ and $n_5 \mid 6$, so $n_5 = 1$ or 6

10 Sylow 3-subgroups give $10 \times 2 = 20$ elements of order 3, since the subgroups intersect trivially pairwise. Similarly, 6 Sylow 5-subgroups give $6 \times 4 = 24$ elements of order 5.

$20 + 24 > 30$, a contradiction, so there can be only one 3-subgroup or one 5-subgroup. A unique *p*-subgroup is normal (conjugate only to itself) and the result follows.

(b) Let N be a unique Sylow *p*-subgroup, where $p = 3$ or $p = 5$. Then the quotient group G/N may be formed, and $|G/N| = 10$ if $p = 3$, $|G/N| = 6$ if $p = 5$. In either case, G/N must be cyclic or dihedral (order is $2 \times$ a prime), and therefore has a normal subgroup, K, of index 2 in G/N, ie $|K| = 5$ if $|G/N| = 10$, $|K| = 3$ if $|G/N| = 6$. *(HB 50 Result 2.1 Groups of order 2p)*

By the Correspondence Theorem, G has a normal subgroup, K' which contains N, such that $K'/N = K$, and $|K'| = |N| \times |K|$. *(HB 37 Theorem 5.2)*

K is the set of cosets of N in K'. There are $|N|$ element of G in each coset, and $|K|$ cosets. The cosets partition K', so $|K'| = |N| \times |K|$.

If $|N| = 3$, $|G/N| = 10$, $|K| = 5$ and $|K'| = 3 \times 5 = 15$
If $|N| = 5$, $|G/N| = 6$, $|K| = 3$ and $|K'| = 5 \times 3 = 15$

This shows that, in either case, G has a normal subgroup of order 15.