M336 1995 Exam Solutions

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<u>Health Warning</u>: This is NOT official OU material. The solutions show how I personally would have tackled the paper. They have not been verified, and it is unlikely that they would agree totally with the Exam Board's answers. (Let me know if you spot any errors)

Part I (55 marks - 90 minutes)

Question 1.

(a) $f = t[\boldsymbol{p}]\lambda[A]$ so $f^{-1} = t[-A^{-1}\boldsymbol{p}]\lambda[A^{-1}]$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ so } A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } \boldsymbol{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{Then} \quad -A^{-1}\boldsymbol{p} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\text{so} \quad f^{-1} = t[(-2,1)]\,\lambda\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(b) Tile types: [3,3,3,3,3,3,3,3], [3,3,3,3,4]
Vertex types: (5,5,8), (5,5,5,5)

(a) HB 5 Rule 5
In 'ordinary' notation:
$\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{p}$
so $\boldsymbol{x} = A^{-1}(\boldsymbol{y} - \boldsymbol{p})$

(b) HB 9-10 **Note:** It doesn't matter what type of brackets you use!

Question 2.

(a) $(r^2s)(r^4s) = r^2(sr^4)s = r^2(r^2s)s = r^4$ (since $s^2 = e$)

(b) $(r^m s)r = r^{m+5}s$ and $r(r^m s) = r^{m+1}s$ and the two are not equal.

(c) r^3

(c) **Note:** $r^3 = r[\pi]$ in D_6

Question 3.

(a) v? yes; h? no; g? yes; so Type 7

(b) Type 2

HB 14 Algorithm

(b) keep v, lose g

Question 4.

Since H has index 2 in G, there are exactly 2 left cosets and exactly 2 right cosets of H.

Consider the coset aH, where $a \in G$. If $a \in H$, then $aH = H = Ha$, (since H is closed). If $a \notin H$, the elements of the coset aH consists of all elements of G which are not in H, since H and aH are disjoint (the two cosets partition G). But these are precisely the elements of Ha, by the same argument. So we must have $aH = Ha$ for all $a \in G$, and H is normal.

HB 17 Lemma 2.1

HB 17 Theorem 2.7

Question 5.

The symmetry group of the colourings is D_9, with order 18. There are 2 rotations of order 3, 6 rotations of order 9, and 9 reflections (of order 2).

g	**cycle type**	**cs(g)**	**number**
e	(1)(2)(3)(4)(5)(6)(7)(8)(9)	x_1^9	1
order 9	(123456789)	x_9	6
order 3	(147)(258)(369)	x_3^3	2
reflections	(1)(29)(38)(47)(56)	$x_1x_2^4$	9

Cycle index: $\frac{1}{18}\left(x_1^9 + 6x_9 + 2x_3^3 + 9x_1x_2^4\right)$

With 2 colours: $\frac{1}{18}\left(2^9 + 6\times 2 + 2\times 2^3 + 9\times 2\times 2^4\right) = 46$

HB p27 Cycle Index Th'm.

Writing down the name of the group of colourings could earn you Brownie points if you go wrong later on.

There are 9 beads, so for $x_a^b x_c^d$ you should have $a\times b + c\times d = 9$.

Coefficients of the elements in the index should add up to $|G| = 18$.

Question 6.

$$
\begin{aligned}
456 &= 1 \times 234 + 222\\
234 &= 1 \times 222 + 12\\
222 &= 18 \times 12 + 6\\
12 &= 2 \times 6
\end{aligned}
$$

So hcf{234, 456} = 6

HB 22 Algorithm

Check:

$236 = 2 \times 3^2 \times 13$

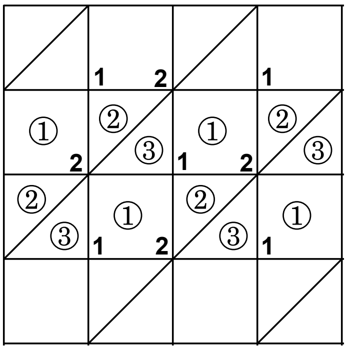
$456 = 2^3 \times 3 \times 19$

so hcf $= 2 \times 3$ ✓

Question 7.

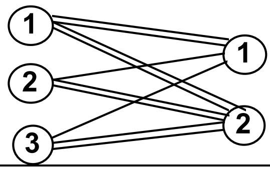
(a) $n_t(\mathfrak{I}) = 3$; $n_v(\mathfrak{I}) = 2$.

(b)



(c)

Tile orbits Vertex Orbits



HB 30 $n_t(\mathfrak{I})$ and $n_v(\mathfrak{I})$

HB 32 Orbits

(c) HB 31 Tile-Vertex Diagram.

Watchpoint:
Make sure your diagram is consistent with the numbering of the orbits in part (b).

Question 8.

(a) $\begin{pmatrix} 2 & 4 & 6 \\ 4 & 10 & 12 \end{pmatrix}$

(b) $\rightarrow \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 0 \end{pmatrix}$ $R_2' = R_2 - 2 \times R_1$

$\rightarrow \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 0 \end{pmatrix}$ $R_1' = R_1 - 2 \times R_2$

$\rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ $C_3' = C_3 - 3 \times C_1$

(c) $A \cong Z_2 \times Z_2 \times Z$

HB 33 Result 1.1

Watchpoint:
The 'extra' column of zeros gives a component Z.
(An extra row is ignored)

Question 9.

(a) $L(\boldsymbol{a}, \boldsymbol{b})$ is a hexagonal lattice.

(b) The transition matrix from $(3\boldsymbol{b} - 2\boldsymbol{a}, 3\boldsymbol{b} - \boldsymbol{a})$ to $(\boldsymbol{a}, 3\boldsymbol{b})$ is

$$\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$$

The matrix has integer entries, and determinant = –1.

So $L(3\boldsymbol{b} - 2\boldsymbol{a}, 3\boldsymbol{b} - \boldsymbol{a}) = L(\boldsymbol{a}, 3\boldsymbol{b})$.

$\boldsymbol{a}' = 3\boldsymbol{b} - 2\boldsymbol{a} = (-\tfrac{1}{2}, \tfrac{3\sqrt{3}}{2})$, $\boldsymbol{b}' = 3\boldsymbol{b} - \boldsymbol{a} = (\tfrac{1}{2}, \tfrac{3\sqrt{3}}{2})$

$\|\boldsymbol{a}'\| = \|\boldsymbol{b}'\| = \sqrt{7}$, and $\boldsymbol{a}' \cdot \boldsymbol{b}' = 26/4$ (so angle not 0 or $\pi/3$)

Therefore $L(3\boldsymbol{b} - 2\boldsymbol{a}, 3\boldsymbol{b} - \boldsymbol{a}) = L(\boldsymbol{a}, 3\boldsymbol{b})$ is rhombic.

(c) $L(\boldsymbol{a}, 2\boldsymbol{b})$ is rectangular.

HB 40 Theorem 5.3 (e)

$\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = 1$, $\boldsymbol{a} \cdot \boldsymbol{b} = \tfrac{1}{2} = \cos\pi/3$

HB 38 Theorem 2.1

Watchpoint: Co-ords of new basis w.r.t. old basis form the **columns** of the matrix.

Watchpoint for part (c):
$\boldsymbol{a} = (1,0)$, $2\boldsymbol{b} = (1, \sqrt{3})$ do not form a *reduced* basis. **Draw a picture** before classifying.

Reduced basis for $L(\boldsymbol{a}, 2\boldsymbol{b})$ is $L(\boldsymbol{a}, 2\boldsymbol{b} - \boldsymbol{a}) = \{(1,0), (0, \sqrt{3})\}$.

Question 10.

$|G| = p^\alpha$, where p is prime and $\alpha > 0$. Suppose there are k orbits under the conjugacy action. Then the class equation for G is:

$$p^\alpha = |\mathrm{Orb}(g_1)| + |\mathrm{Orb}(g_2)| + \ldots + |\mathrm{Orb}(g_k)| \quad (*)$$

By the Orbit-Stabiliser theorem, $|\mathrm{Orb}(g_i)|$ divides $|G|$, so each element in the sum is 1 or a positive power of p. $|\mathrm{Orb}(e)| = 1$.

The centre is the union of single element conjugacy classes.

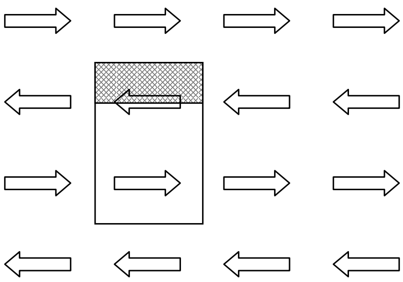
Suppose G has a trivial centre. Then each element in the sum in (*), apart from $|\mathrm{Orb}(e)|$, would be a positive power of p, so that the right hand side of (*) would be congruent to 1 modulo p. But $p \mid p^\alpha$, so the left side is congruent to 0 modulo p, a contradiction. Hence G must have a non-trivial centre.

HB 37 Centre
HB 36 Class equation

This was proved in exercise 5.1 in Unit GR4.

Question 11.

(a)



(b) p2mg

HB 45 Algorithm

Check:
highest order? 2
reflections? yes
2 directions? no

Part II A (Geometry)

Question 12.

(a)

g	**cycle type**	**cs(g)**
e	(1)(2)…(8)	x_1^8
r, r^3, r^5, r^7	(12345678)	x_8
r^2, r^6	(1357)(2468)	x_4^2
r^4	(15)(26)(37)(48)	x_2^4

Cycle index: $\frac{1}{8}(x_1^8 + 4x_8 + 2x_4^2 + x_2^4)$

(b) Pattern inventory:

$$\tfrac{1}{8}[(P+Y)^8 + 4(P^8+Y^8) + 2(P^4+Y^4)^2 + (P^2+Y^2)^4]$$

i) No yellow faces: Coefficient of P^8:

$\frac{1}{8}[1 + 4 + 2 + 1] = 1$

ii) 1 yellow face: Coefficient of P^7Y in $(P+Y)^8$:

$\frac{1}{8}[8] = 1$

iii) 2 yellow faces: Coefficients of P^6Y^2 in $(P+Y)^8$ & $(P^2+Y^2)^4$

$$\tfrac{1}{8}\left[\binom{8}{6} + 4\right] = \tfrac{1}{8}[28 + 4] = 4$$

(c) Total: $\frac{1}{8}[2^8 + 4\times 2 + 2\times 2^2 + 2^4] = 36$.

HB 27 Theorem 4.2

(b) For pattern inventory, replace x_k^m by $\left(P^k + Y^k\right)^m$ in the cycle index in (a).

Note ii): all powers in terms other than the first are even
Note iii): powers in other terms are multiples of 4

(c) For total number, replace x_k^m by 2^m in the cycle index in (a)

Question 13.

(a)

Highest order of rotation?	2
Any reflections?	Yes
Reflections in 2 directions?	Yes
Type of reflections?	Rhombic (alternating glides & reflections)

Therefore type is **c2mm**

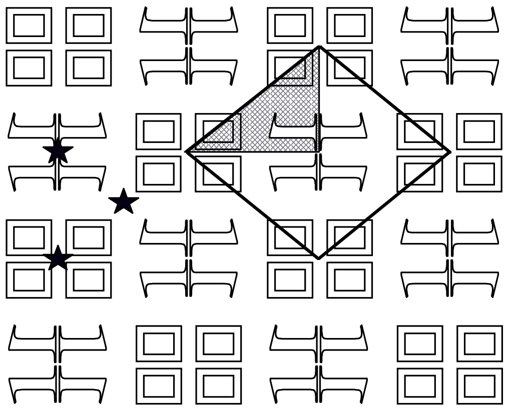
HB 45 Algorithm

HB 44 Theorem 2.7

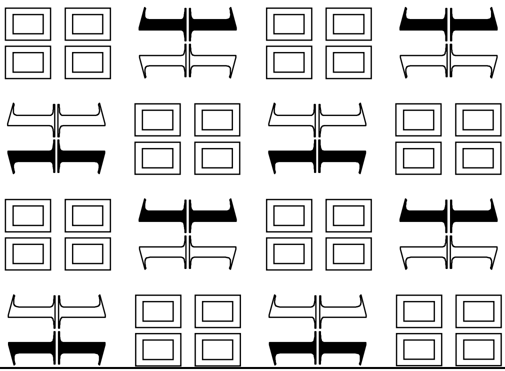
(Cont. on next page)

Question 13 continued:

(b)/(c)



(d)



Watchpoint (b): Orientation of the basic parallelogram.

Choose an origin, find the points which the origin can be **translated** to, then choose the nearest to construct the basic parallelogram. It is a rhombus, consistent with the type of reflections in part (a).

Note (d): The shading kills off the horizontal reflections, but keeps the vertical ones. So there are no longer reflections in two directions.

Watchpoint: Make sure the shading retains the order 2 rotation, otherwise you end up with type cm. This is why the shading differs in alternate rows.

Question 14.

(a) $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \|\boldsymbol{c}\| = 2$, $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{c} = \boldsymbol{a}\cdot\boldsymbol{c} = 0$
ie the basis consists of 3 orthogonal vectors of equal length.
Therefore the lattice is **primitive cubic**.

(b) $\|\boldsymbol{a}\|=\|\boldsymbol{b}\|= \sqrt{10}$, angle $\neq \pi/2$ or $\pi/3$, so $L(\boldsymbol{a}, \boldsymbol{b})$ is rhombic
$\boldsymbol{c} = (0, 0, 2) + \boldsymbol{a} + \boldsymbol{b}$, so offset = (0, 0)
Therefore the lattice is **base-centred orthorhombic.**

(c) $L(\boldsymbol{a}, \boldsymbol{b})$ is square
$\boldsymbol{c} = (0, 0, \sqrt{7}) + \sqrt{5}\boldsymbol{a} + \sqrt{6}\boldsymbol{b}$, so offset = $(\sqrt{5}-2, \sqrt{6}-2)$
Therefore the lattice is **triclinic**

(d) $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = 10$, $\boldsymbol{a}\cdot\boldsymbol{b} = 0$, so $L(\boldsymbol{a}, \boldsymbol{b})$ is square
$\boldsymbol{c} = (0, 0, 5) + \tfrac{1}{2}\boldsymbol{a} + \tfrac{1}{2}\boldsymbol{b}$, so offset = $(\tfrac{1}{2}, \tfrac{1}{2})$
Vertical separation = $5 = \tfrac{1}{2}\|\boldsymbol{a}\|$
Therefore the lattice is **body-centred cubic.**

(e) $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = 2$, $\boldsymbol{a}\cdot\boldsymbol{b} = 2 = \tfrac{1}{2}\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|$, so $L(\boldsymbol{a}, \boldsymbol{b})$ is hexagonal
$\boldsymbol{c} = (0, 0, 1) + \boldsymbol{a} + \boldsymbol{b}$, so offset = (0, 0)
Therefore the lattice is **hexagonal.**

HB 56 Result 3.2
HB 40 Theorem 5.3

Note (*a*): $L(\boldsymbol{a}, \boldsymbol{b})$ is square in the plane $x = y$. $\boldsymbol{c}$ lies in the plane $x = -y$ (orthog. to $x = y$), offset = (0, 0) w.r.t. $\boldsymbol{a}, \boldsymbol{b}$, and separation = $\|\boldsymbol{a}\|$.

Note (b): $(\boldsymbol{a}, \boldsymbol{b})$ is not a *reduced* basis, but HB 40 Theorem 5.3 (c) says L is rhombic if there is *a* basis (not necessarily reduced) with $\|\boldsymbol{a}\|=\|\boldsymbol{b}\|$, but $\boldsymbol{a}, \boldsymbol{b}$ not orthogonal.

Note (e): HB 40 Th'm 5.3 (e)
$\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta$,
and $\cos \pi/3 = \tfrac{1}{2}$

Part II B (Groups)

Question 15.

(a) For all $x \in H \cap K$, $x \in H$ and $x \in K$, and for all $g \in G$,
$gxg^{-1} \in H$ (H is normal) and $gxg^{-1} \in K$ (K is normal)
So $gxg^{-1} \in H \cap K$, and $H \cap K$ is normal in G.

(b) Let $G = D_6$, $H = \{e, s\}$ and $K = \{e, rs\}$, then neither H nor K is normal in G (eg $rsr^{-1} = r^2s \notin H$, $r(rs)r^{-1} = r^3s \notin K$)
But $H \cap K = \{e\}$ is (trivially) normal in G. **Cont on next page.**

Watchpoint: The question **states** that $H \cap K$ is a subgroup of G, so no need to prove it. If it had said "Prove that $H \cap K$ is a normal subgroup of G", you would need to prove it was a subgroup first, before proving normality.

Question 15 continued:

(c) Yes. Let $G = D_6$, $H = K = \{e, s\}$.
Then $H \cap K = \{e, s\}$ is (trivially) normal in both H and K.
But $\{e, s\}$ is not normal in G (eg $rsr^{-1} = r^2s \notin \{e, s\}$).

(c) Look for the simplest examples – you don't need anything fancy.

Question 16.

(a) $|G| = 72 = 2^3 \times 3^2$

prime	**factors**	**label**
3	3^2	3a
	3×3	3b
2	2^3	2a
	2×2^2	2b
	$2 \times 2 \times 2$	2c

label	***p*-primary form**	**canonical form**
2a3a:	$Z_8 \times Z_9$	Z_{72}
2b3a:	$Z_2 \times Z_4 \times Z_9$	$Z_2 \times Z_{36}$
2c3a:	$Z_2 \times Z_2 \times Z_2 \times Z_9$	$Z_2 \times Z_2 \times Z_{18}$
2a3b:	$Z_8 \times Z_3 \times Z_3$	$Z_3 \times Z_{24}$
2b3b:	$Z_2 \times Z_4 \times Z_3 \times Z_3$	$Z_6 \times Z_{12}$
2c3b:	$Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_3$	$Z_2 \times Z_6 \times Z_6$

(b) Only $Z_8 \times Z_9 \cong Z_{72}$ and $Z_8 \times Z_3 \times Z_3 \cong Z_3 \times Z_{24}$ have **every** subgroup of order 4 cyclic (since the 2-primary components are cyclic). The other groups all have order 4 subgroups isomorphic to $Z_2 \times Z_2$.

HB 33 canonical form

HB 36 *p*-primary form

Question 17.

(a) $|G| = 2^3 \cdot 5 \cdot 11$. Let n_p be the number of Sylow p-subgroups of G. Then

$n_{11} \equiv 1 \pmod{11}$ and $n_{11} \mid 40$, so $n_{11} = 1$.
$n_5 \equiv 1 \pmod 5$ and $n_5 \mid 88$, so $n_5 = 1$ or 11.
$n_2 \equiv 1 \pmod 2$ and $n_2 \mid 55$, so $n_2 = 1, 5, 11$ or 55.

(b) Since $n_{11} = 1$, the Sylow 11-subgroup is unique, hence normal in G. Let this subgroup be N; then $|N| = 11$, and G has a normal subgroup of order 11.

The quotient group G/N may be formed, and $|G/N| = 40 = 2^3 \cdot 5$
By theorem 5.1 (*G*R5), G/N has subgroups of orders 2, 4, 5 and 8 (being the prime powers that divide $|G/N|$).

By the Correspondence Theorem, these subgroups give rise to corresponding subgroups of G that contain N, with orders $2\times 11 = 22$, $4\times 11 = 44$, $5\times 11 = 55$ and $8\times 11 = 88$ respectively.

(c) By part (a), $n_5 = 1$ or 11. If $n_5 = 1$, the Sylow 5-subgroup is unique, hence normal in G, and has order 5.

If $n_5 = 11$, there are 11 conjugate Sylow 5-subgroups, $H_1, \ldots, H_{11}$, of order 5. Under the conjugacy action $|\mathrm{Orb}(H_i)| = 11$, and by the Orbit-Stabiliser Theorem, $|\mathrm{Stab}(H_i)| = |G|/11 = 40$

$\mathrm{Stab}(H_i)$ is a subgroup of G, so G has a subgroup of order 40.

HB 48 Theorem 3.1 (Summary of the Sylow results).

HB 49 Theorem 5.1 (Prime power subgroups)

HB 37 Correspondence Th'm

Note: If K is a subgroup of G/N, then $K = H/N$, where H is the corresponding subgroup of G containing N ie K is the set of cosets of N in H.

If $|K| = 5$, there are 5 cosets of N in $K = H/N$. Since there are 11 elements of G in each coset, and the cosets partition H, then there must be $11\times 5 = 55$ elements of G in H ie $|H| = 55$.

HB 27 Orbit-Stabiliser Th'm