M336 1996 Exam Solutions

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Part I (55 marks - 90 minutes)

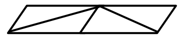
Question 1.

(a) Tile types: [3, 4, 3, 6], [3, 6, 4,6]
Vertex types: (4, 4, 4), (4, 4, 4, 4), (4, 4, 4, 4, 4, 4)

(b) (i) 1
(ii) 2 eg $(x, y) \to (2x + y, y)$

HB 9-10

(i) 

(ii) 

Question 2.

(a) $sr^3s^{-1} = r^{-3}ss^{-1} = r^5$ (since $r^8 = e$)

(b) $sr^m = r^{-m}s = r^ms$ if r^m commutes with s, so $m \equiv -m \pmod 8$,
ie $m = 0$ or $m = 4$, and only e and r^4 commute with s.

Question 3.

(a) v? no; h? no; g? no; r? yes, so Type 5

(b) Type 1

HB 14 Algorithm
(b) lose r, don't gain anything

Question 4.

Let $x, y\ (\neq e) \in G$, so $x^{-1} = x$ and $y^{-1} = y$ (both order 2).
Then $xy = x^{-1}y^{-1} = (yx)^{-1} = yx$ (as yx has order 2). Hence G is Abelian.

Question 5.

(a) The symmetry group is $V = \{e, r, v, h\}$

g	**cycle type**	**cs(g)**
e	(A)(B)(C)(D)	x_1^4
r, v, h	(AC)(BD)	x_2^2

Cycle index: $\frac14\left(x_1^4 + 3x_2^2\right)$

(b) Let AB = P, BC = Q, CD = R, DA = S

g	**cycle type**	**cs(g)**
e	(P)(Q)(R)(S)	x_1^4
r	(PR)(QS)	x_2^2
v, h	(QS)	$x_1^2x_2$

Cycle index: $\frac14\left(x_1^4 + x_2^2 + 2x_1^2x_2\right)$

(c) $\frac14\left[(R+G)^4 + \left(R^2+G^2\right)^2 + 2(R+G)^2\left(R^2+G^2\right)\right]$

HB p27 Cycle Index Th'm.

Writing down the name of the symmetry group could earn you Brownie points if you go wrong later on.

(c) Replace x_k^m in (b) by $\left(R^k + G^k\right)^m$

Question 6.

$\langle(1, 0)\rangle$, $\langle(0, 2)\rangle$, $\langle(1, 2)\rangle$, $\langle(1, 4)\rangle$, $\langle(1, 6)\rangle$, $\langle(2, 2)\rangle$,

$a \in Z_4$, $b \in Z_8$, $(a,b) \in Z_4 \times Z_8$
$|(a,b)| = \text{lcm}(|a|, |b|)$
NB: 2 order 4 elements in each

Question 7.

(a) $n_v(\Im) = 2$.

(b) Tile type is [4,8,8], so $n_v(\Im) = 2 = n_t(\Im)[\frac14 + \frac18 + \frac18] = \frac12 n_t(\Im)$.
So $n_t(\Im) = 4$.

(c) There are 12 lines in the diagram (8 from 1st vertex orbit, 4 from 2nd), and 2 lines enter each edge orbit.
So $n_e(\Im) = {}^{12}/_2 = 6$.

HB 29 [$n_t(\Im)$ and $n_v(\Im)$]

HB 32 Vertex-Edge Diagram

Question 8.

HB 33 -341

(a) $\begin{bmatrix} 2 & 8 & 4 \\ 4 & 6 & 8 \\ 2 & 2 & 6 \end{bmatrix}$

(b) $\rightarrow \begin{bmatrix} 2 & 8 & 4 \\ 0 & -10 & 0 \\ 0 & -6 & 2 \end{bmatrix}$ $\quad R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - R_1$

$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & -6 & 2 \end{bmatrix}$ $\quad C_2 \rightarrow C_2 - 4C_1$, $C_3 \rightarrow C_3 - 2C_1$

$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad R_2 \rightarrow -R_2$, $C_2 \rightarrow C_2 + 3C_3$

(c) $A \cong Z_2 \times Z_2 \times Z_{10}$

Question 9.

Suppose $L(\boldsymbol{a}, \boldsymbol{b})$ has a rotation r of order 3 about the vertex $n\boldsymbol{a} + m\boldsymbol{b}$, where $r = r\left[\pm 2\pi/3\right]$. Since $-n\boldsymbol{a} - m\boldsymbol{b}$ is also a lattice point, rotation $s = r[\pi]$ about the origin, and hence about any lattice point, is also a symmetry of L.

Then $sr = r\left[\pi \pm 2\pi/3\right] = r\left[\mp \pi/3\right]$ of order 6 is a symmetry of L (closure).

HB 4 (Isometry Toolkit)

Question 10.

HB 37 Centre

$Z(G)$ is a (normal) subgroup of G, so $|Z(G)|$ divides $|G| = 50$ (Lagrange). Divisors of 50 are 1, 2, 5, 10, 25 and 50. G is not Abelian, so $|Z(G)| \neq 50$.

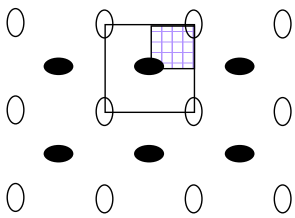
HB 37 Theorem 4.3

If $|Z(G)| = 25$, then $|G/Z(G)| = 2$, a prime, so $G/Z(G)$ is cyclic. This would make G Abelian (by Th'm 4.3), a contradiction, so $|Z(G)| \neq 25$. Similarly, $|Z(G)| \neq 10$, since the quotient group (order 5) would again be cyclic.

Hence $|Z(G)|$ is at most 5.

Question 11.

(a)



(b) p2mm

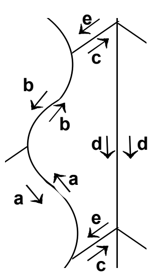
HB 45 Algorithm

Check:
highest order? 2
reflections? yes
2 directions? yes
type? rectangular

Part II A (Geometry)

Question 12

(a)



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Question 12 (continued)

(b) $a^{\rightarrow}\ b^{\rightarrow}\ c^{\rightarrow}\ d^{\rightarrow}\ e^{\rightarrow}$
$a^{\leftarrow}\ b^{\leftarrow}\ e^{\leftarrow}\ d^{\rightarrow}\ c^{\leftarrow}$

(c) 5

(d) 4
Edge ***d***: same, similarly directed, so stabiliser is $\{e, h\}$
Edge ***a***: same, oppositely directed, so stabiliser is $\{e, r\}$

(e) (i) There is only 1 edge ***a*** which is mapped to ***b***, so ***b*** is mapped to ***a***. ***c*** can't also be mapped to ***a***.
(ii) The first edge ***a*** must be mapped to the adjacent edge ***a*** by a rotation about the centre of the tile. All edges would therefore rotate, so they should all be labelled ***a***.

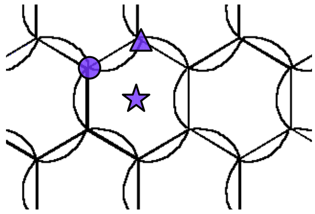
HB 31 (incidence symbol)

(c) no tile is mapped to itself under Γ, so no two edge sides have the same label.

(d) ***c*** ↔ ***e*** under translation. HB 30 Theorem 5.2 to classify stabilisers.

Question 13.

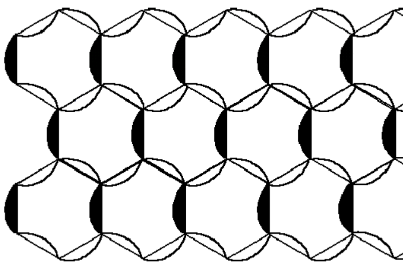
(a) 3



(b) 2

(c) highest order of rotation? 3
any reflections? yes
all 3-centres on refl axes? no (only the one at the centre)
Therefore p31m

(d)



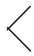
(e) cm

(f) p6mm

(b) ▲ ↔ ● under reflection

(c) HB 45 Algorithm

(d) lose all rotations, and 2 of the 3 reflection exes

(e) reflection in horizontal axis is rhombic (use basis  with O at centre of the tile)

(f) rotations are now order 6

Question 14.

(a) All direct symmetries are rotations:

$\Gamma^+(Q_1) = C_4$ [2 × order 4, 1 × order2 (vertical axis)]

$\Gamma^+(Q_2) = A_4$ [3 × order 2 (axes through centres of faces) 8 × order 3 (axes through vertices)]

$\Gamma^+(Q_3) = D_4$ [2 × order 4, 1 × order 2 (vertical axis), 4 × order 2 (axes through centres of vertical faces and edges)]

$\Gamma^+(Q_4) = D_2$ [3 × order 2 (axis through centres of faces)]

$\Gamma^+(Q_5) = C_3$ [2 × order 3 (axis through 1 pair of vertices)]

(b) Q_3 only

(c) Q_1 only

(d) $\Gamma(Q_1) \cong C_4 \times C_2$ (since $\sigma_0 \in \Gamma$)
$\Gamma(Q_5) \cong D_3$ ($\sigma_0 \notin \Gamma$)

Note: Full marks for correct groups without justification.

Q_2 - fit a tetrahedron inside.

(b) 1 mark for correct cube, 1 mark for no others
ditto for (c)

(d) HB 52 classification

Part II B (Groups)

Question 15.

(a) $a(bcb^{-1})bd = abc(b^{-1}b)d$ (where $b^{-1}b = e$)
$= abcd$

(b) Closure: Let $h_1k_1, h_2k_2 \in HK$, where $h_i \in H$, $k_i \in K$. Then $k_ih_jk_i^{-1} = h' \in H$ (since H is normal in G), so

$$h_1k_1h_2k_2 = h_1k_1h_2\left(k_1^{-1}k_1\right)k_2 = h_1\left(k_1h_2k_1^{-1}\right)k_1k_2 = h_1h'k_1k_2 \in HK$$

since H and K are closed.

Inverse: $(hk)^{-1} = k^{-1}h^{-1}$ in G, and

$$k^{-1}h^{-1} = k^{-1}h^{-1}\left(kk^{-1}\right) = \left(k^{-1}h^{-1}k\right)k^{-1} = h''k^{-1} \in HK,$$

by the subgroup properties of H and K, and normality of H.

Identity: e is the identity of G.
$e \in H$ and $e \in K$, (subgroups), so $ee = e \in HK$..

Hence HK is a subgroup of G.

(c) (i) $G = C_6 = \langle a\colon a^6 = e\rangle$, $H = \{e, a^2, a^4\}$, $K = \{e, a^3\}$
(ii) $G = D_6 = \langle r, s\colon r^6 = s^2 = e,\ sr = r^5s\rangle$, $H = \{e, r, r^2, r^3, r^4, r^5\}$,
$K = \{e, r^3, s, r^3s\}$ and $H \cap K = \{e, r^3\}$

(b) Use the trick they reminded you of in (a).

(c) (i) C_6 is Abelian, so all subgroups are normal.

(c) (ii) H has index 2, so is normal.

Question 16.

(a) $|G| = 1000 = 2^3 \times 5^3$

prime	factors	label
5	5^3	5a
	5×5^2	5b
	$5 \times 5 \times 5$	5c
2	2^3	2a
	2×2^2	2b
	$2 \times 2 \times 2$	2c

label	p-primary form	canonical form
2a5a:	$Z_8 \times Z_{125}$	Z_{1000}
2b5a:	$Z_2 \times Z_4 \times Z_{125}$	$Z_2 \times Z_{500}$
2c5a:	$Z_2 \times Z_2 \times Z_2 \times Z_{125}$	$Z_2 \times Z_2 \times Z_{250}$
2a5b:	$Z_8 \times Z_5 \times Z_{25}$	$Z_5 \times Z_{200}$
2b5b:	$Z_2 \times Z_4 \times Z_5 \times Z_{25}$	$Z_{10} \times Z_{100}$
2c5b:	$Z_2 \times Z_2 \times Z_2 \times Z_5 \times Z_{25}$	$Z_2 \times Z_{10} \times Z_{50}$
2a5c:	$Z_8 \times Z_5 \times Z_5 \times Z_5$	$Z_5 \times Z_5 \times Z_{40}$
2b5c:	$Z_2 \times Z_4 \times Z_5 \times Z_5 \times Z_5$	$Z_5 \times Z_{10} \times Z_{20}$
2c5c:	$Z_2 \times Z_2 \times Z_2 \times Z_5 \times Z_5 \times Z_5$	$Z_{10} \times Z_{10} \times Z_{10}$

HB 33 canonical form

HB 36 p-primary form

(b) Groups with 5-primary component $Z_5 \times Z_{25}$ or $Z_5 \times Z_5 \times Z_5$ (ie the last 6 groups) clearly have a subgroup isomorphic to $Z_5 \times Z_5$.

Groups with 5-primary component equal to Z_{125} (ie the first 3 groups) have only cyclic subgroups of order 25, whereas $Z_5 \times Z_5$ is not cyclic.

Question 17.

(a) $|G| = 2^2 p$. Let n_q be the number of Sylow q-subgroups of G. Then

$n_p \equiv 1 \pmod{p}$ and $n_p \mid 4$, so $n_p = 1$ (since $p \neq 3$).

Let K be the Sylow p-subgroup. Then K is unique, hence normal in G, $|K| = p$, and K is cyclic hence Abelian.

$n_2 \equiv 1 \pmod{2}$ and $n_2 \mid p$, so $n_2 = 1$ or p (since p is odd).

If $n_2 = 1$, let H be the (unique hence normal) Sylow 2-subgroup. Then $|H| = 4$, and H ($\cong Z_2 \times Z_2$ or Z_4) is Abelian.

Since $K \cap H = \{e\}$, the conditions for Theorem 4.1 would hold, so that $G \cong K \times H$, and G would be Abelian, a contradiction. Therefore, there are p Sylow 2-subgroups.

(b) K is normal in G, so the quotient group G/K may be formed, and $|G/K| = 4$. Thus $G/K \cong Z_2 \times Z_2$ or $G/K \cong Z_4$ and in either case, has a normal subgroup N, of order 2 (index = 2).

By the Correspondence Theorem, N gives rise to a corresponding normal subgroup of G containing K, with order $2|K| = 2p$.

(c) The p Sylow 2-subgroups, $H_1, \ldots, H_p$, (of order 4) are all conjugate in G. Under the conjugacy action $|\mathrm{Orb}(H_i)| = p$ for $i = 1, 2, \ldots, p$, and by the Orbit-Stabiliser Theorem, $|\mathrm{Stab}(H_i)| = |G| / |\mathrm{Orb}(H_i)| = 4p/p = 4$.

For any $h \in H_i$, $h \wedge H_i = hH_ih^{-1} = H_i$ (since H_i is closed), so that $H_i \subseteq \mathrm{Stab}(H_i)$. Since $|H_i| = |\mathrm{Stab}(H_i)| = 4$, we must have $H_i = \mathrm{Stab}(H_i)$ for each $i = 1, 2, \ldots, p$.

(a) HB 48 Theorem 3.1 (Summary of the Sylow results).

HB 49 Theorem 4.1 (Direct products)
[Alternatively, you could quote Result 1.1, HB 49]

(b) HB 37 Correspondence Th'm

Note: If N is a subgroup of G/K, then $N = N'/K$, where N' is the corresponding subgroup of G containing K ie N is the set of cosets of K in N'.

If $|N| = 2$, there are 2 cosets of K in $N = N'/K$. Since there are p elements of G in each coset, and the cosets partition N', there must be $2p$ elements of G in N'

(c) HB48 Th'm 3.1 (c) [all Sylow p-subgroups are conjugate].
HB 27 Orbit-Stabiliser Th'm