M336 1997 Exam Solutions

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Health Warning: This is NOT official OU material. The solutions show how I personally would have tackled the paper. They have not been verified, and it is unlikely that they would agree totally with the Exam Board's answers. (Let me know if you spot any errors)

Part I (55 marks - 90 minutes)

Question 1.

(a) $f = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then

$$f^2 = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\left(\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

HB 9-10 **Note:** It doesn't matter what type of brackets you use!

(b) Tile types: [4, 3, 4, 3, 4, 3, 4, 3], [3, 4, 3, 4]
Vertex types: (4, 4, 8), (4, 4, 4, 4), (4, 8, 4, 8)

Question 2.

(a) $(r^4s)(r^3s) = r^4(sr^3)s$ (associative)
$= r^4r^{-3}ss$ $(sr = r^5s = r^{-1}s)$
$= rs^2$
$= r^4$ $(s^2 = r^3)$

(b) $(r^4s)^{-1} = s^{-1}r^{-4}$
$= s^3r^{-4}$ $(s^4 = e)$
$= s^2(sr^{-4})$
$= r^3(r^4s)$ $(s^2 = r^3,\ sr^{-4} = r^4s)$
$= rs$ $(r^7 = rr^6 = r)$

Question 3.

(a) v? yes; h? no; g? yes; so Type 7

(b) v? no; h? no; g? yes; so Type 4

HB 14 Algorithm

Question 4.

<u>*Closure*</u>: Let $h_1n_1,\ h_2n_2 \in HN$ (where $h_1, h_2 \in H, n_1,\ n_2 \in N$). Then
$(h_1n_1)(h_2n_2) = h_1(n_1h_2)n_2$
$= h_1(h_2h_2^{-1}n_1h_2)n_2$
$= h_1h_2n_3n_2$ $(n_3 = h_2^{-1}n_1h_2 \in N,$ normal)
$\in HN$ (since H and N are closed)

<u>*Identity*</u>: Identity of G is e; $e \in H$, $e \in N$ (subgroups)
Therefore $e = ee \in HN$.

<u>*Inverses*</u>: $(hn)^{-1} = n^{-1}h^{-1}$ in G
$n^{-1}h^{-1} = h^{-1}hn^{-1}h^{-1} = h^{-1}n'$ $(n' = hn^{-1}h^{-1} \in N,$ normal)
$\in HN$

Note the old trick of adding $h_2h_2^{-1}$ in the middle to make use of the fact that N is normal.

This is a standard method to change nh to hn'.

Question 5.

(a)

g	**cycle type**	**cs(g)**	**number**
e	(1)(2)(3)(4)	x_1^4	1
f, g	(123)(4)	$x_1 x_3$	2
r, s, t	(1)(2)(34)	$x_1^2 x_2$	3

Cycle index: $\frac{1}{6}\left(x_1^4 + 2x_1x_3 + 3x_1^2x_2\right)$

(b) $\frac{1}{6}\left((B+Y)^4 + 2(B+Y)\left(B^3+Y^3\right) + 3(B+Y)^2\left(B^2+Y^2\right)\right)$

(c) Equivalence classes: $\frac{1}{6}\left(2^4 + 2\times 2\times 2 + 3\times 2^2 \times 2\right) = 8$

HB p27 Cycle Index Th'm. There are 4 triangles, so for $x_a^b x_c^d$, $a\times b + c\times d = 4$. Coefficients of the elements in the index should add up to $|G| = 6$..

(b) Replace x_k^m in (a) by $\left(B^k + Y^k\right)^m$

(c) Replace x_k^m in (a) by 2^m

Question 6.

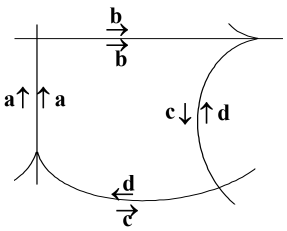
(a)
$138 = 1\times 102 + 36$
$102 = 2\times 36 + 30$
$36 = 1\times 30 + 6$
$30 = 5\times 6$ So hcf{102, 138} = 6

(b)
$6 = 36 - 30$
$= 36 - (102 - 2\times 36)$
$= 3\times 36 - 102$
$= 3\times(138 - 102) - 102$
$= 3\times 138 - 4\times 102$

HB 22 Algorithm
Check:
$138 = 2 \times 3 \times 23$
$102 = 2 \times 3 \times 17$
so hcf = 2×3✓

Question 7.

(a)



(b) $\vec{a}\ \vec{b}\ \vec{c}\ \vec{d}$
$\vec{a}\ \vec{b}\ \overleftarrow{d}\ \overleftarrow{c}$

(c) Edge ***a***: Same and similarly directed; $\{e, h\}$

(d) Edge ***c***: Different, directed; $\{e\}$

(b) HB 31 Incidence symbol

(c) and (d) HB 30 Th'm 5.2, Classification of edge stabilisers.

Question 8.

(a) $\begin{bmatrix} 2 & 4 & 6 \\ 10 & 22 & 48 \\ 6 & 20 & 48 \end{bmatrix}$

(b) $\rightarrow \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 18 \\ 0 & 8 & 30 \end{bmatrix}$ $\quad R_2' = R_2 - 5\times R_1$, $R_3' = R_3 - 3\times R_1$

$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 18 \\ 0 & 8 & 30 \end{bmatrix}$ $\quad C_2' = C_2 - 2\times C_1$, $C_3' = C_3 - 3\times C_1$

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(b) **Watchpoint:**
Show your working. It's easy to make arithmetic errors, so you need to show what you were trying to do, even if you get it wrong.

Question 8 cont.

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 18 \\ 0 & 0 & -42 \end{bmatrix} \quad R_3' = R_3 - 4 \times R_2$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 42 \end{bmatrix} \quad \begin{array}{l} C_3' = C_3 - 9 \times C_2 \\ C_3' = -C_3 \end{array}$$

(c) $A \cong Z_2 \times Z_2 \times Z_{42}$

Question 9.

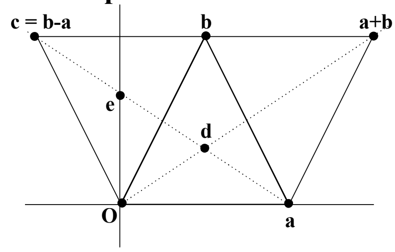
(a) i) Vector of D: ⅓ (**a** + **b**)

ii) Vector of E: ⅓ (**c** + **b**) = ⅓ (**b** – **a** + **b**) = ⅓ (2**b** – **a**)

(b) $q[\pi/3]$ (reflection in the line OB)

(c) $t[½\mathbf{a}]q[½\mathbf{b}, 0]$ (reflect in horizontal through ½**b**, translate through ½**a**.)

Draw a picture.



Question 10.

(a) $|G| = 108 = 2^2 \times 3^3$

prime	factors	p-primary component
2	2^2	Z_4
	2×2	$Z_2 \times Z_2$
3	3^3	Z_{27}
	3×3^2	$Z_3 \times Z_9$
	$3 \times 3 \times 3$	$Z_3 \times Z_3 \times Z_3$

There are 6 possible groups

(b) Groups with a *cyclic* subgroup of order 9 must have 3-primary components containing Z_{27} or Z_9. These are:

$Z_4 \times Z_{27} \cong Z_{108}$
$Z_2 \times Z_2 \times Z_{27} \cong Z_2 \times Z_{54}$
$Z_4 \times Z_3 \times Z_9 \cong Z_3 \times Z_{36}$
$Z_2 \times Z_2 \times Z_3 \times Z_9 \cong Z_6 \times Z_{18}$

Question 11.

Highest order of rotation? 3
Any reflections? Yes
All 3-centres on reflection axes? Yes
Therefore type p3m1.

HB 45 Algorithm

Part II A (Geometry)

Question 12.

(a) Identity (e).
Rotation (r) through π about the centre of the card (no turning over).
2 rotations (s, t) through π about the diagonals, turning the card over.

(b)

g	cycle type	cs(g)
e	(1)(2)…(6)	x_1^6
r	(13)(46)(2)(4)	$x_1^2 x_2^2$
s, t	(14)(25)(36)	x_2^3

Cycle index: $\frac{1}{4}(x_1^6 + x_1^2 x_2^2 + 2x_2^3)$ **continued on next page**

(b) There are 6 rectangles to deal with, so for $x_a^b x_c^d$ you should have $a \times b + c \times d = 6$.

Coefficients of the elements in the index should add up to $|G| = 4$.

Question 12 cont.

(c) Equivalence classes: ¼ ($3^6 + 3^2 \times 3^2 + 2\times 3^3$) = 216

(c) Replace x_a^b in (b) by 3^b

(d) Using the Orbit-Stabiliser theorem, we can establish that the number of orbits equals $\frac{1}{|G|}\sum_{x\in X}|\mathrm{Stab}(x)|$ (*).

Arrange the inert pairs in a table, where the columns correspond to the elements of X, and the rows correspond to elements of G. The number of entries in the column headed x_i is $|\mathrm{Stab}(x_i)|$, and the number of entries in the row labelled g_j is $|\mathrm{Fix}(g_j)|$.

The total number of inert pairs is then $\sum_{x\in X}|\mathrm{Stab}(x)| = \sum_{g\in G}|\mathrm{Fix}(g)|$.

This, together with (*) above, proves the Counting Lemma.

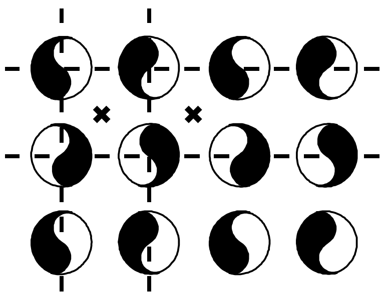
(d) HB 27 Corollary 3.1 and Theorem 3.1.

Question 13.

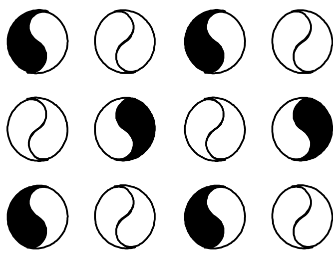
(a) Highest order of rotation? 2
Any reflections? No
Any glide reflections? Yes
Therefore type is **p2gg**

HB 45 Algorithm

(b) & (c)



(d)



(c) **Watchpoint:**.
Check that your axes are consistent with your answers in part (a) - ie no reflections; all indirect symmetries are glides, so dotted lines

(d) **Watchpoint:**
Make sure you don't kill off the rotations with your new shading.

Question 14.

(a)(i) Let ***a*' = *c*, *b*' = *b*, *c*' = *a*.**

Then $\|\mathbf{a}'\| = \|\mathbf{b}'\| = 2$, and $\mathbf{a}' \cdot \mathbf{b}' = 0$, so $L(\mathbf{a}', \mathbf{b}')$ is square.
The transition matrix from ***a*', *b*', *c*'** to ***a*, *b*, *c***

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

has integer entries, and determinant = –1. So $L(\mathbf{a}', \mathbf{b}', \mathbf{c}') = L(\mathbf{a}, \mathbf{b}, \mathbf{c})$ by Theorem 1.2.

(ii) $\mathbf{c}' = (3, 0, 0) + \frac{1}{2}\mathbf{a}'$, where (3, 0, 0) is orthogonal to ***a*'** and ***b*'**.
Thus the offset is (½, 0)
So the Bravais type is base-centred orthorhombic.

(ii) HB56 Result 3.2
Note: It was vital to change the basis first, before applying result 3.2. It would have been difficult to establish the 2-d lattice in the plane containing the original ***a*** and ***b***.

(b) $\|\mathbf{d}\| = \|\mathbf{e}\| = \|\mathbf{f}\| = \sqrt{8}$, $\mathbf{d} \cdot \mathbf{e} = \mathbf{e} \cdot \mathbf{f} = \mathbf{d} \cdot \mathbf{f} = 0$, so the basis consists of orthogonal vectors of equal length, and $L(\mathbf{d}, \mathbf{e}, \mathbf{f})$ is Primitive Cubic.

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Question 14 (cont) (c) The direct symmetry group is D_4 and $\|D_4\| = 8$. **g** / **cycle type** / **cs(g)** / **number** e — (1)(2)…(10) — x_1^{10} — 1 r, r^3 — (9)(10)(1234)(5678) — $x_1^2x_4^2$ — 2 r^2 — (9)(10)(13)(24)(57)(68) — $x_1^2x_2^4$ — 1 $r^m s$ — (17)(28)(35)(46)(910) — x_2^5 — 4 Cycle index: $^1/_8\,(x_1^{10} + 2x_1^2x_4^2 + x_1^2x_2^4 + 4x_2^5)$ So the number of equivalence classes for 2 colours is: $^1/_8\,(2^{10} + 2.2^2.2^2 + 2^2.2^4 + 4.2^5) = 156$	(c) D_4 is the symmetry group of the glued face of one of the cubes. Reflections in axes of the glued face correspond with rotations through π about the reflection axes. ***Note:*** there are 10 faces to consider separately, so make sure that the sub- and super-scripts in cs(g) reflect this: ie for $x_a^b x_c^d$, you should have a×b + c×d = 10

Part II B (Groups)

Question 15 (a) <u>Subgroup</u> *Closure:* Let $x, y \in Z(G)$, so that $xg = gx$ and $yg = gy$ for all $g \in G$. Then $xyg = xgy = gxy$, so $xy \in Z(G)$. *Identity:* $eg = ge$ for all $g \in G$, so $e \in Z(G)$. *Inverse:* Let $x \in Z(G)$, so that $xg = gx$ for all $g \in G$. Then $x^{-1}g = (g^{-1}x)^{-1} = (xg^{-1})^{-1} = gx^{-1}$ (since $x \in Z(G)$). So $x^{-1} \in Z(G)$. <u>Normality</u> Let $x \in Z(G)$, so that $xg = gx$ for all $g \in G$. Then $gxg^{-1} = xgg^{-1} = x \in Z(G)$. So $Z(G)$ is normal in G. (b) Suppose $A = G/Z(G)$ is cyclic. Then there is an element $a \in G$, such that $aZ(G)$ generates A; ie $A = \{aZ(G), a^2Z(G), ..., a^mZ(G) = Z(G)\}$ Since the cosets partition G, this shows that every element in G is of the form $a^s z_i$ for some $z_i \in Z(G)$. Let $x = a^s z_i$, $y = a^t z_j$, $x, y \in G$. Then $xy = a^s z_i a^t z_j = a^s a^t z_i z_j = a^t a^s z_j z_i = a^t z_j a^s z_j = yx$. So G is Abelian. (c) Let $G = S_3$, $N = A_3$ (even permutations). Then $\|G/N\| = 2$, a prime. Thus G/N is cyclic.	(a) ***Note:*** it is absolutely vital that you prove $Z(G)$ is a **subgroup** (5 marks) before you prove normality (2 marks).
Question 16. (a) $X = \{xH: x \in G\} = \{x_1H, x_2H, ..., x_kH\}$ where $x_1 = e$. *i)* $g \wedge xH = gxH \in X$, since $gx \in G$. *ii)* $e \wedge xH = exH = xH$ since $ex = x$ iii) $gh \wedge xH = ghxH = g(hxH) = g \wedge (hxH) = g \wedge (h \wedge xH)$. (b) The action defines a homomorphism ψ from G to the group of bijections $\Gamma(X)$ from X to X. $\Gamma(X)$ is isomorphic to S_k, (by associating the *label* i of the element x_iH with the *number* i in the set {1, 2, ..., k}). Composing the homomorphism ψ with the isomorphism $\Gamma(X) \to S_k$ gives the required homomorphism $\phi: G \to S_k$.	**continued on next page**

Question 16 (continued) (c) We know that Ker(ϕ) is a normal subgroup of G by th'm 4.4, unit IB4. Suppose $k \in$ Ker(ϕ). Then $k \wedge xH = kxH = xH,$ for all $x \in G$. In particular, taking $x = e,$ we have $kH = keH = eH = H,$ so $k \in H$. Therefore Ker(ϕ) $\subseteq H$. (d) In this case, Ker(ϕ) must be trivially normal, ie Ker(ϕ) = $\{e\}$, or Ker(ϕ) is the whole of G. Since Ker(ϕ) $\subseteq H$, and H is not the whole of G, we must have Ker(ϕ) = $\{e\}$, so G is *isomorphic* to Im(ϕ), a subgroup of S_k. By Lagrange's theorem, the order of a subgroup divides the order (k!) of S_k, and the result follows.	(d) In M203, there is a theorem that says $\lvert$Ker(ϕ)$\rvert \times \lvert$Im(ϕ)$\rvert = \lvert G \rvert$. In this case, since $\lvert$Ker(ϕ)$\rvert$ = 1 we have $\lvert$Im(ϕ)$\rvert = \lvert G \rvert$. Since Im(ϕ) is a subgroup of the co-domain S_k, the result follows from Lagrange's theorem. Unfortunately, the theorem is not in M336, so you'd have to prove it before you could use it.
Question 17. (a) $\lvert G \rvert = 3^2 \cdot 5$. Let n_p be the number of Sylow p-subgroups of G. Then $n_3 \equiv 1 \pmod 3$ and $n_3 \mid 5$, so $n_3 = 1$. $n_5 \equiv 1 \pmod 5$ and $n_5 \mid 9$, so $n_5 = 1$ (b) By (a), G has a unique Sylow subgroup corresponding to each prime divisor of $\lvert G \rvert = 45$. By Result 1.1 of Unit GR6, G may be expressed as the internal direct product of its Sylow subgroups. Let H be the Sylow 3-subgroup. Then $\lvert H \rvert = 9$ $(= 3^2)$, so either $H \cong Z_9$ or $H \cong Z_3 \times Z_3$. Let K be the Sylow 5-subgroup. Then $\lvert K \rvert = 5$, a prime, so $K \cong Z_5$. Since $G \cong H \times K$ we must have $G \cong Z_9 \times Z_5$ or $G \cong Z_3 \times Z_3 \times Z_5$. In either case, G is Abelian.	(a) HB 48 Theorem 3.1 (Summary of the Sylow results). (b) HB 49 Result 1.1. HB 37 Result 5.2: A group of order p^2 is Abelian (p prime)