M336 1998 Exam Solutions

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Health Warning: This is NOT official OU material. The solutions show how I personally would have tackled the paper. They have not been verified, and it is unlikely that they would agree totally with the Exam Board's answers. (Let me know if you spot any errors)

Part I (55 marks - 90 minutes)

Question 1.

(a) Tile types: [3,3,4,3,3,4], [3, 3,4,4], [3,3,3,3,3,3]
Vertex types: (4, 4, 6,6), (4, 6,6), (6,6,6)

(b) No (triangles have 6 edges but only 3 sides)

(c) (4,6,6,) [*or any of the others – they all have 2 orbits!*]

a) HB 9/10 for definitions of tile/vertex types.

b) HB 8 for definition of edge–to–edge. Explanation not actually needed, but it can't do any harm to add it

Question 2.

(a)
$$\begin{aligned}(r^2s)^2 &= r^2sr^2s\\ &= r^2srrs\\ &= r^2r^3srs && (sr = r^3s)\\ &= rr^3ss && (r^4 = e,\ \ sr = r^3s)\\ &= r^2 && (r^4 = e,\ \ s^2 = r^2)\end{aligned}$$

(b)
$$\begin{aligned}(r^2s)^2 &= r^2\\ (r^2s)^3 &= r^2r^2s = s \quad (r^4 = e)\\ (r^2s)^4 &= (r^2)^2 = r^4 = e.\end{aligned}$$
Since $r^2 \neq e$ and $s \neq e$, order = 4

(a) *Alternative method:*

$(r^2s)^2 = (s^3)^2 = s^6 = s^2 = r^2$

(since $s^2 = r^2$, $s^4 = e$)

Question 3.

(a) v? yes; h? no; g? no; so Type 2

(b) v? yes; h? no; g? yes; so Type 7

HB 14 Algorithm

Question 4.

$H \cap N$ is an intersection of H and N, which are themselves subgroups of G. Therefore $H \cap N$ is a subgroup of H (and N) by theorem 2.3.

For all $n \in N$, we have:

- $gng^{-1} \in N$ for all $g \in G$ (N is normal in G)
- $hnh^{-1} \in N$ for all $h \in H$ (since $h \in G$) (**)

Let $a \in H \cap N$, so $a \in N$, and $a \in H$.

Then, for all $h \in H$, $hah^{-1} \in N$ (by **) and $hah^{-1} \in H$ (since H is closed), so $hah^{-1} \in H \cap N$, and $H \cap N$ is normal in H.

NB: prove a subgroup *first,* then prove normality.

Thm 2.3: HB p 16.

Question 5.

(a) $G \cong D_3$, so $|G| = 6$.

g	**cycle type**	**cs(g)**	**number**
e	(A)…(F)	x_1^6	1
r, r^2	(AEC)(BDF)	x_3^2	2
s, rs, r^2s	(AB)(CF)(DE)	x_2^3	3

Cycle index: $\frac{1}{6}\left(x_1^6 + 2x_3^2 + 3x_2^3\right)$

(a) and (b)
HB p27 Cycle Index Th'm. Brownie points for stating what G is.

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Question 5 (continued)

(b) Relabel edges AB = 1, …, FA = 6

g	cycle type	cs(g)	number
e	(1)…(6)	x_1^6	1
r, r^2	(135)(246)	x_3^2	2
s, rs, r^2s	(1)(4)(26)(35)	$x_1^2 x_2^2$	3

Cycle index: $\frac{1}{6}\left(x_1^6 + 2x_3^2 + 3x_1^2x_2^2\right)$

(c) Equivalence classes: $\frac{1}{6}\left(2^6 + 2\times 2^2 + 3\times 2^2 \times 2^2\right) = 20$

(c) Replace x_k^m in (b) by 2^m

Question 6.

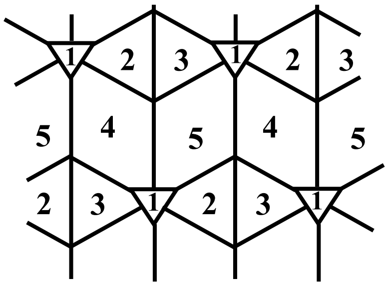
(a) $\langle(0, 3)\rangle$, $\langle(15, 0)\rangle$, $\langle(15, 3)\rangle$, $\langle(15, 6)\rangle$

(b) 2 in each subgroup, so 8 in all

Order 3 elements:
in Z_{45}: 15, 30
in Z_9: 3, 6

Question 7.

(a)



(b) $n_v(\Im) = 8$

(c) $n_e(\Im) = n_t(\Im) + n_v(\Im)$ (Euler). So $n_e(\Im) = 13$.

(b) 6 orbits of degree–3 vertices (of the triangle) + 2 orbits of degree–4 vertices.

(c) Euler's theorem: HB p29

Question 8.

(a) $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 6 & 6 \\ 2 & 12 & 14 \end{bmatrix}$

(b) $\rightarrow \begin{bmatrix} 2 & 0 & 2 \\ 0 & 6 & 6 \\ 0 & 12 & 12 \end{bmatrix}$ $R_3' = R_3 - R_1$

$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 6 \\ 0 & 12 & 12 \end{bmatrix}$ $C_3' = C_3 - C_1$

$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3' = R_3 - 2 \times R_2$

$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $C_3' = C_3 - C_2$

(c) $A \cong Z_2 \times Z_6 \times Z$ so has a subgroup isomorphic to Z. Hence A is infinite

(b) **Watchpoint:**
Show your working. It's easy to make arithmetic errors, so you need to show what you were trying to do, even if you get it wrong.

Question 9.

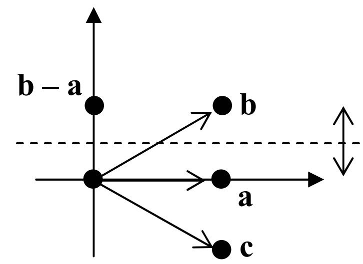
(a) Reduced basis (**a'**, **b'**) = {(0,1), (2,0)} where **a' = b – a, b' = a**

(b) L(**a, b**) = L(**a', b'**) is rectangular (**a'.b'** = 0, |**a'**| < |**b'**|)

(c) |**b**| = |**c**| = √5 **b.c** = 3 ≠ 0 or ½ |**b**|^2. So L(**b, c**) is rhombic.

(d) q[(2, 0), (1, ½), 0] = q[½(**b**+**c**), ½**b**, 0]

Draw a picture.



Question 10.

(a) { (12)(34), (13)(24), (14)(23) }

(b) *Closure*: (12)(34)(13)(24) = (14)(23)
(12)(34)(14)(23) = (13)(24)
(13)(24)(14)(23) = (12)(34)
The elements commute with each other, so the set is closed.

Inverse: Each element has order 2, so is a self–inverse

Identity: e belongs to the set

Therefore the set is a subgroup.

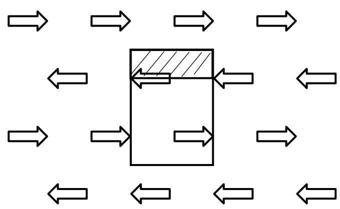
Since {e} is a conjugacy class, the group is the union of 2 conjugacy classes, and is therefore normal (by Theorem 4.4).

(b) NB: prove it's a subgroup first, then that it's normal. (It's isomorphic to the Klein group; all non–trivial elements have order 2)

Theorem 4.4: HB p37

Question 11.

(a)



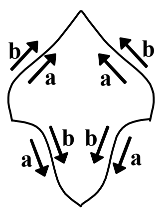
(b) Highest order of rotation? 2
Any reflections? Yes
2 directions? No
Therefore type $p2mg$.

(b)
HB p45 Algorithm
Note: you don't *need* the detail, but it's probably worth putting it in, if you have time, in case you go down the wrong path.

Part II A (Geometry)

Question 12

(a)



(b) $a^{\rightarrow}$ $a^{\leftarrow}$ $b^{\rightarrow}$ $b^{\leftarrow}$
$b^{\rightarrow}$ $b^{\leftarrow}$ $a^{\rightarrow}$ $a^{\leftarrow}$

(a) The tile stabiliser contains the identity and a vertical reflection.

(b) HB p31 for definition of tile symbol

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Question 12 (continued)

(c) 2 edge side orbits

(d) 1 edge orbit

(e) $n_3 = 2$, $n_6 = 1$

(f) $n_e(\mathfrak{I}) = \frac{1}{2}(3n_3 + 6n_6) = 6$

(g) (i) *b* is undirected, so there is a **reflection** that fixes *b*. Therefore *a* and *c* should also be undirected or appear twice.

(ii) There's a rotation mapping the first edge to the third, since *a* appears twice with the same direction. The rotation would therefore map the 2nd edge to the fourth, ie the edges would be labelled *abab*, all in the same direction.

(c) *Edge side orbits* are orbits of edges **within** a particular tile, ie under the symmetries of the *tile stabiliser*. *a*'s and *b*'s are in different orbits when you're considering only the symmetries in the tile stabiliser.

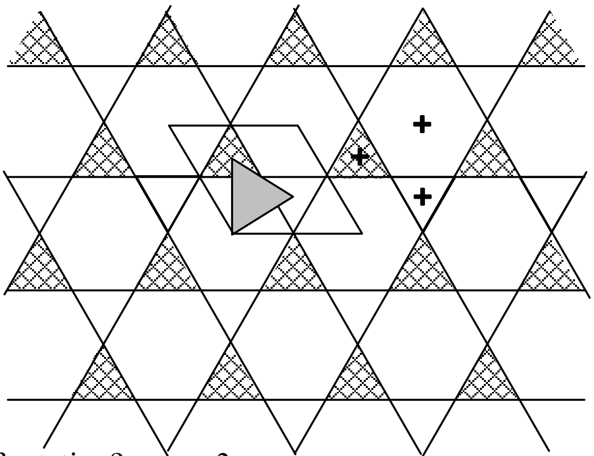
(d) *Edge orbits* are the orbits of edges under the *full* symmetry group (any edge maps to any other edge under the symmetries of the group)

(f) Formula: HB p29 Theorem 3.1

Question 13.

(a) 3 orbits of rotation centres

(c)



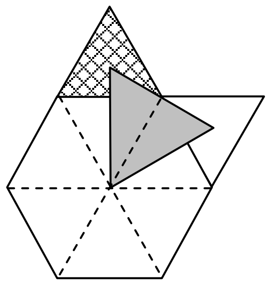
(b) Highest order of rotation? 3
Any reflections? Yes
All 3–centres on reflection axes? Yes
Therefore type is ***p*3*m*1**

(d) A hexagon is made up of 6 unit equilateral triangles.

So the height of the generating equilateral triangle is 1

Therefore the side of the generating triangle is

$1/\sin(\pi/3) = 2/\sqrt{3}$



b) HB p45 (Algorithm)

c) You weren't asked for the basic parallelogram, but it's much easier to find a generating region if you draw a basic parallelogram first.

(d) *Alternative method:*
The distance from the centre of a triangle to the corner is $\sqrt{3}/3 = 1/\sqrt{3}$
Since a side of the generating region goes from the centre of a shaded triangle to the centre of an unshaded triangle, its length must be $2/\sqrt{3}$.

Question 14. (a) (i) $L(\boldsymbol{a}, \boldsymbol{b})$ is rectangular $\boldsymbol{c} = (0, 0, 2) + \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$, so offset = (½, ½) Therefore $L(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is body centred orthorhombic (ii) $\lVert\boldsymbol{a}\rVert = \lVert\boldsymbol{b}\rVert = 2$, $\boldsymbol{a}\cdot\boldsymbol{b} = 2 = \frac{1}{2}\lVert\boldsymbol{a}\rVert\,\lVert\boldsymbol{b}\rVert$, so $L(\boldsymbol{a}, \boldsymbol{b})$ is hexagonal $\boldsymbol{c} = (0, 0, 1) + \boldsymbol{a} + \boldsymbol{b}$, so offset = (0, 0) Therefore the lattice is **hexagonal.** (iii) $L(\boldsymbol{a}, \boldsymbol{b})$ is square $\boldsymbol{c} = (0, 0, \sqrt{7}) + \sqrt{5}\boldsymbol{a} + \sqrt{6}\boldsymbol{b}$, so offset = $(\sqrt{5}-2, \sqrt{6}-2)$ Therefore the lattice is **triclinic**	(a) HB p56 Result 3.2 HB p40 Theorem 5.3 NB: Take care with this type of question. Make sure you're using a suitable basis for the 2–d lattice before you find the offset. '*Suitable*' means reduced except possibly for rhombic lattices, where you want a basis of equal vectors (which may not be a reduced basis).
(b) (i) The direct symmetry group is A_4 and $\lvert A_4\rvert = 12$. **g** / **cycle type** / **cs(g)** / **number** e / (1)(2)…(12) / x_1^{12} / 1 $r[2\pi/3]$ / (123)(456)(789)(10,11,12) / x_3^{4} / 8 $r[\pi]$ / (13)(25)(46)(7,11)(8,10)(9,12) / x_2^{6} / 3 Cycle index: $\frac{1}{12}\left(x_1^{12} + 8x_3^{4} + 3x_2^{6}\right)$ (ii) With 2 colours, the number of equivalence classes is: $\frac{1}{12}\left(2^{12} + 8 \times 2^4 + 3.2^6\right) = 368$ (iii) Let x be the number of rotational classes indistinguishable from their mirror images, and y be the number which *can* be distinguished. Under the full symmetry group, each class would consist of either *one* rotational class which is its own mirror image, or *two* rotational classes which are (distinguishable) mirror images of each other. So $x + y = 368$ and $x + \frac{1}{2}y = 218$, giving $x = 68$	(b) (i) Brownie points for writing down the name of the symmetry group. (ii) Replace x_k^m by 2^m in cycle index. (iii) If the mirror image of *every* rotation class were distinguishable, then under the full symmetry group there would be 368/2 = 184 classes; ie each rotation class + its mirror image would now be in a single class.
Part II B (Groups)	
Question 15 (a) *Closure:* Let $x, y \in L$, so that $xH = Hx$ and $yH = Hy$ Then $xyH = xHy = Hxy$, so $xy \in L$. *Identity:* $eH = He$ so $e \in L$. *Inverse:* Let $x \in L$, so that $xH = Hx$. Then $x^{-1}(xH)x^{-1} = x^{-1}(Hx)\,x^{-1}$. So $Hx^{-1} = x^{-1}H$, and $x^{-1} \in L$. So L is a subgroup of G $hH = Hh$ (=H) for all $h \in H$, so $H \subseteq L$, and H is a group. Hence H is a subgroup of L. For all $x \in L$, $xH = Hx$ by definition of L So H is a normal subgroup of L. (b) Since H is normal in K, $kH = Hk$ for all $k \in K$, so $K \subseteq L$. K is a group, hence K is a subgroup of L. (c) (i) $x\{e, (12)\} = \{e, (12)\}x$ if and only if x commutes with (12). Elements with this property are e and (12) itself, so $L = \{e, (12)\}$ (ii) $H = \{e, (123), (132)\}$ is normal in S_3 (index 2) Therefore $xH = Hx$ for all $x \in S_3$ and $L = S_3$	(a) ***Note:*** it is absolutely vital that you prove H is a **subgroup** (3 marks) before you prove normality (2 marks).

Question 16.

(a) $|G| = 1575 = 3^2 \times 5^2 \times 7$

prime	factors	label
7	7	7a
5	5^2	5a
	5×5	5b
3	3^2	3a
	3×3	3b

label	*p*-primary form	canonical form
3a5a7a:	$Z_9 \times Z_{25} \times Z_7$	Z_{1575}
3b5a7a:	$Z_3 \times Z_3 \times Z_{25} \times Z_7$	$Z_3 \times Z_{525}$
3a5b7a:	$Z_9 \times Z_5 \times Z_5 \times Z_7$	$Z_5 \times Z_{315}$
3b5b7a:	$Z_3 \times Z_3 \times Z_5 \times Z_5 \times Z_7$	$Z_{15} \times Z_{105}$

(b) $Z_{45} \cong Z_9 \times Z_5$ so any group containing a subgroup isomorphic to Z_{45} must also have subgroups isomorphic to Z_9 and Z_5. It is clear from the p–primary form that the only such groups are Z_{1575} and $Z_5 \times Z_{315}$.

(a) Canonical decomposition HB p33 Theorem 3.2
p–primary decomposition HB p38

Question 17.

(a) $|G| = 2^3 \cdot 3^2$. Let n_3 be the number of Sylow 3-subgroups of G.
Then $n_3 \equiv 1 \pmod 3$ and $n_3 \mid 8$, so $n_3 = 1$ or 4.

(b) Group action of G on X: $g \wedge X_i = gX_i g^{-1}$ for all $g \in G$, $i = 1, 2, 3, 4$.

Closure: Sylow subgroups are conjugate to each other.
So $g \wedge X_i = gX_i g^{-1} = X_j \in X$

Identity: $e \wedge X_i = eX_i e^{-1} = X_i$

Composition: $gh \wedge X_I = ghX_i (gh)^{-1}$
$= ghX_i h^{-1}g^{-1} = g(h \wedge X_i)g^{-1} = g \wedge (h \wedge X_i)$

The properties of Lemma 5.1 are therefore satisfied, thus there is a homomorphism $\psi: G \to \Gamma(X)$ (definition of a group action) and $\Gamma(X)$ is isomorphic to S_4. Then $\phi: G \to S_4$ is the composition of ψ and the isomorphism from $\Gamma(X)$ to S_4.

(c) If $\text{Ker}(\phi) = G$, then each X_i would be conjugate only to itself, which contradicts Sylow's 3rd Th'm (Sylow p–subgroups conjugate to each other).

If $\text{Ker}(\phi) = \{e\}$, then $G \cong \text{Im}(\phi) \le S_4$, but $|G| = 72$ and $|S_4| = 24$, so this is clearly impossible.

(d) If G has 1 Sylow 3-subgroup then that subgroup is normal and non–trivial.

If G has 4 Sylow 3-subgroups then $\text{Ker}(\phi)$ is a normal subgroup of G, and is non–trivial by part (c).

Therefore G cannot be simple.

(a) Theorem 3.1 (Summary of the Sylow results) HB p48

(b) Definition of group action HB p11 (Glossary)

Definition of homomorphism HB p50 (Glossary)

(c) Sylow: Result 3.1, HB p48
$\text{Im}(\phi) \le S_4$: Th'm 4.4 (b) HB p17
First Isomorphism Th'm HB p18

(d) Kernel is normal
Thm 4.4 HB p17
Simple group HB p49