M336 1999 Exam Solutions

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<u>**Health Warning**</u>: **This is NOT official OU material. The solutions show how I personally would have tackled the paper. They have not been verified, and it is unlikely that they would agree totally with the Exam Board's answers.** (Let me know if you spot any errors)

Part I (55 marks - 90 minutes)

Question 1.

(a) $f = t[\mathbf{p}]\lambda[A] \ \Rightarrow\ f^{-1} = t[-A^{-1}\mathbf{p}]\lambda[A^{-1}]$

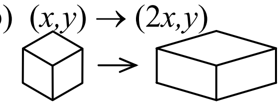
$A^{-1} = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$ so $A^{-1}\mathbf{p} = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (2,-1)$

so $f^{-1} = t[(-2,1)]\lambda\begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$

(b) 2 non–congruent tile shapes

(c) maximum of 3 non–congruent tile shapes

(b) $(x,y) \to (2x,y)$



2 of the tiles remain congruent

(c) Affine Congruency Lemma (HB p4, Lemma 4.1) The quadrilaterals are in 3 orientations; none at angle 0 or π to any other.

Question 2.

(a) $(r^4s)(r^3s) = r^4r^3ss = r$

(b) $(r^4s)^{-1} = s^{-1}r^{-4} = sr^2 = r^{-2}s = r^4s$

(c) $sr = r^5s \neq rs$, since $r^5 \neq r$. So not all elements commute.

(a) $sr^3 = r^3s,\ s^2 = e,\ r^6 = e$

(b) Alternatively, $(r^4s)^{-1} = r^4s$ (order 2)

Question 3.

(a) v? yes; h? no; g? no; so Type 2

(b) $\Gamma(F_4) = \langle g_{1/2} \rangle$, since $(g_{1/2})^2 = t$, so $g_{1/2}$ generates the group.

Alternatively: $\Gamma(F_4) \cong \Gamma(F_1)$, which is generated by a single element.

(a) HB 14 Algorithm

(b) Th'm 4.1, HBp13

Question 4.

(a) Let $G = D_6,\ H = \{e, rs\},\ J = \{e, s\}$

Then $HJ = \{e, rs, s, r\}$ is not a group, since not closed. ($r^2 \notin HJ$)

(b) If $HJ = JH$, then for any $jh \in JH$, $jh \in HJ$, so $jh = h'j'$ for some $h' \in H,\ j' \in J$.

Closure: $h_1j_1,\ h_2j_2 \in HJ$, then $h_1j_1\,h_2j_2 = h_1(h'j')j_2 = (h_1h')(j'j_2) \in HJ$. (By closure of H and J).

Identity: $e \in H, e \in J$, so $ee = e \in HJ$.

Inverse: $hj \in HJ$, then $(hj)^{-1} = j^{-1}h^{-1} \in JH = HJ$.

Note: in part (a), $JH = \{e, s, rs, r^5\} \neq HJ$.

Question 5.

(a) $G \cong D_4$, so $|G| = 8$.

g	**cycle type**	**cs(g)**	**number**
e	(1)(2)(3)(4)	x_1^4	1
r, r^3	(1234)	x_4	2
r^2, rs, r^3s	(13)(24)	x_2^2	3
s, r^2s	(1)(3)(24)	$x_1^2x_2$	2

Cycle index: $\frac{1}{8}\left(x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2x_2\right)$

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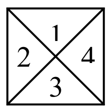
(a) Draw a picture:



HB p27 Cycle Index Th'm.

NB: 4 triangles, so for $x_a^bx_c^d$ must have $a\times b + c\times d = 4$.

Coefficients of the elements in the index should add up to $|G| = 8$.

Question 5 (continued)

(b) For 3 colours (including colourings with all the same colour), there are:

$$\tfrac{1}{8}\left(3^4 + 2\times 3 + 3\times 3^2 + 2\times 3^2 \times 3\right) = 21$$

Knock off the three colourings with only one colour, to give the number of distinguishable colourings with at least 2 colours as 18

(b) Replace x_k^m in (a) by 3^m

Question 6.

(a) $245 = 1\times231 + 14$

$231 = 16\times14 + 7$

$14 = 2\times7 + 0$ so hcf(245, 231) = 7

(b) $7 = 231 - 16\times14$

$= 231 - 16\times(245 - 231)$

$= 17\times231 - 16\times245$

(c) lcm(245, 231) = (245 × 231) / hcf(245, 231) = 8085

Euclidean algorithm HB p22

(a) Check: $245 = 5 \times 7^2$

$231 = 3 \times 7 \times 11$

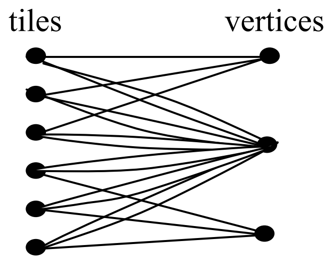
(b) Easy check: use your calculator on the final sum to make sure it does actually equal 7. (It's surprising how many people don't bother.)

(c) Theorem 2.2, HB p22

Question 7.

(a) $n_v(\Im) = 3$, $n_t(\Im) = 6$

(b)



(c) $n_e(\Im) = n_t(\Im) + n_v(\Im)$ (Euler). So $n_e(\Im) = 9$.

(a) vertices: 1 of degree 12, and 2 of degree 3.
tiles: 6 different orientations of the triangle.

(c) Euler's theorem: HB p29

Question 8.

(a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 5 & 5 \\ 6 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 5 \\ 0 & 0 & 12 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 6R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 12 \end{bmatrix} \quad C_3 \to C_3 - C_2$$

(b) $A \cong Z_1 \times Z_5 \times Z_{12} \cong Z_5 \times Z_{12} \cong Z_{60}$ (Z_1 is trivial, 5, 12 are coprime)

(c) $Z_5 \times Z_{12} \cong Z_5 \times Z_3 \times Z_4$, represented by

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) **Watchpoint:**
Show your working. It's easy to make arithmetic errors, so you need to show what you were trying to do, even if you get it wrong.

(b) Canonical form Th'm 3.2 HB p33.

(c) Alternatives:

$$\begin{bmatrix} 5 & 0 \\ 0 & 12 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 60 \end{bmatrix}$$

Question 9.

(a) Hexagonal

(b) Let $\boldsymbol{a'} = \mathbf{1}(3\boldsymbol{b}) + \mathbf{2}(\boldsymbol{a}) = \left(-\tfrac{1}{2}, \tfrac{3\sqrt{3}}{2}\right)$, $\boldsymbol{b'} = \mathbf{1}(3\boldsymbol{b}) + \mathbf{1}(\boldsymbol{a}) = \left(\tfrac{1}{2}, \tfrac{3\sqrt{3}}{2}\right)$

The transition matrix from $(\boldsymbol{a'}, \boldsymbol{b'})$ to $(\boldsymbol{a}, 3\boldsymbol{b})$ is

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ with integer entries and determinant 1.

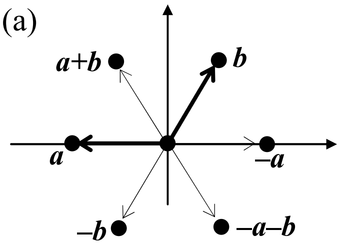
Therefore $L(\boldsymbol{a}, 3\boldsymbol{b}) = L(\boldsymbol{a'}, \boldsymbol{b'})$.

$|\boldsymbol{a'}| = |\boldsymbol{b'}| = \sqrt{7}$ and $\boldsymbol{a'} \bullet \boldsymbol{b'} = {}^{26}/_{4} \neq 0$ or $\pm\tfrac{1}{2}|\boldsymbol{a'}|^2$, so $L(\boldsymbol{a'}, \boldsymbol{b'})$ is rhombic, hence $L(\boldsymbol{a}, 3\boldsymbol{b})$ is rhombic.

(c) $L(\boldsymbol{a}, 2\boldsymbol{b})$ is rectangular.

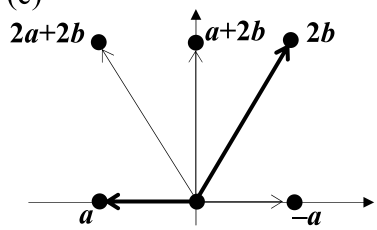
Pictures, with lots of vertices.

(a)



(b) Identity of lattices: Th'm 1.2 HB p38

(c)



Question 10.

(a) $|G| = 30$, and $Z(G)$ is a normal subgroup of G, so $|Z(G)|$ divides 30, and $|Z(G)| \in \{1, 2, 3, 5, 6, 10, 15, 30\}$.

$|Z(G)| \neq 30$, since G is non–Abelian.

If $|Z(G)| \in \{6, 10, 15\}$, then $|G/Z(G)| \in \{5, 3, 2\}$, so the quotient group would be cyclic (prime order), and G would be Abelian – a contradiction. Therefore $|Z(G)| \leq 5$.

(b) $|S_3 \times Z_5| = 30$, so $|Z(S_3 \times Z_5)|$ is *at most* 5 by part (a). (*)

$\{e\} \times Z_5$ is a subgroup of $Z(S_3 \times Z_5)$, since its elements commute with every element of $S_3 \times Z_5$, and $|\{e\} \times Z_5| = 5$.

So $|Z(S_3 \times Z_5)|$ is *at least* 5. (**)

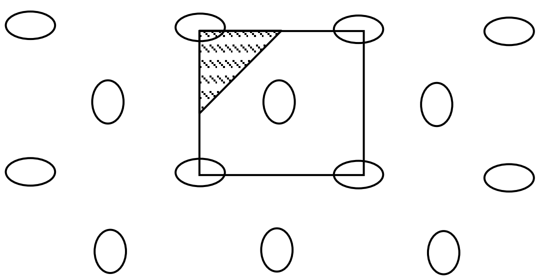
Hence, by (*) and (**) $|Z(S_3 \times Z_5)| = 5$ [and $Z(S_3 \times Z_5) \cong \{e\} \times Z_5$].

(a) Th'm 4.3, HB p37.

(b) If $(x, y) \in Z(S_3 \times Z_5)$, then x commutes with every element of S_3, (so $x = e$) and y commutes with every element of Z_5, which is Abelian.

Question 11.

(a)



(b) Highest order of rotation? 4
Any reflections? Yes
4 directions? Yes
Therefore type *p*4*mm*.

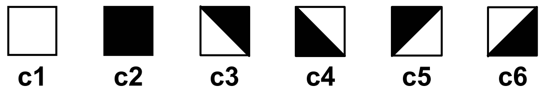
Note: Do part (b) first, then you know that the point group is D_4.

(a) Since $|D_4| = 8$, the area of the generating region is $^1/_8 \times$ the area of the basic parallelogram. [Unit GE4, p8 final paragraph.]

(b) HB p45 Algorithm
Note: you don't *need* the detail, but it's probably worth putting it in, if you have time, in case you go down the wrong path.

Question 12

(a) There are 6 possible colourings for each quadrant:



Each quadrant may be coloured independently, so the total number of colourings is $6^4 = 1296$.

(b) $G = \{e, r, r^2, r^3\}$ and $|G| = 4$.

$\text{fix}(e) = 6^4 = 1296$ [e fixes all colourings]

$\text{fix}(r) = \text{fix}(r^3) = 6$ [only one quadrant coloured independently]

$\text{fix}(r^2) = 6^2 = 36$ [two quadrants coloured independently]

Hence number of colourings = $\frac{1}{4}(1296 + 2\times6 + 36) = 336$.

(c) Now $|G| = |D_4| = 8$

Rotations are the same as for part (b).

Let s be reflection (ie turning over) in the horizontal axis.

$\text{fix}(s) = \text{fix}(r^2s) = 6^2 = 36$ [two quadrants coloured independently]

$\text{fix}(rs) = \text{fix}(r^3s) = 6\times4^2 = 96$

[6 possibilities for the pair of opposite quadrants that map to each other, 4 possibilities each for the quadrants that map to themselves]

Hence number of colourings = $\frac{1}{8}(1296 + 2\times6 + 36 + 2\times36 + 2\times96) = 201$

(d) For unpartitioned quadrants using 6 colours, there are still 6 ways to colour each quadrant. Without turning over, fix(g) is the same as in part (b):

$\text{fix}(e) = 6^4$

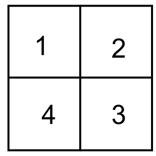
$\text{fix}(r) = \text{fix}(r^3) = 6$ (all quadrants the same colour)

$\text{fix}(r^2) = 6^2$ (opposite quadrants the same colour).

If the square can be turned over, $\text{fix}(s) = \text{fix}(r^2s) = 6^2$ remain the same, since adjacent pairs of quadrants must have the same colour.

But this time, the quadrants that map to themselves on reflection in the diagonals may be coloured in *any* of the 6 colours

So $\text{fix}(rs) = \text{fix}(r^3s) = 6^3 = 216$, compared with $6\times4^2 = 96$ in part (c).

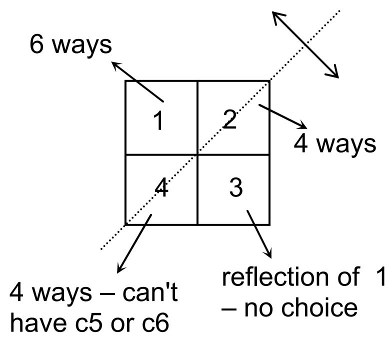


(b) r, r^3: colour 1 quadrant, (6 possibilities), others are rotations of the first, so no choice. 4–cycle eg (1234)

r^2: colour 2 adjacent quadrants independently (6^2 possibilities), then the other pair is a rotation of the first pair through π, so no choice. Two 2–cycles eg (13)(24)

(c) s, r^2s: colour 2 adjacent quadrants independently (6^2 possibilities), then the other pair is reflection of the first pair, so no choice.
Two 2–cycles eg (14)(23)

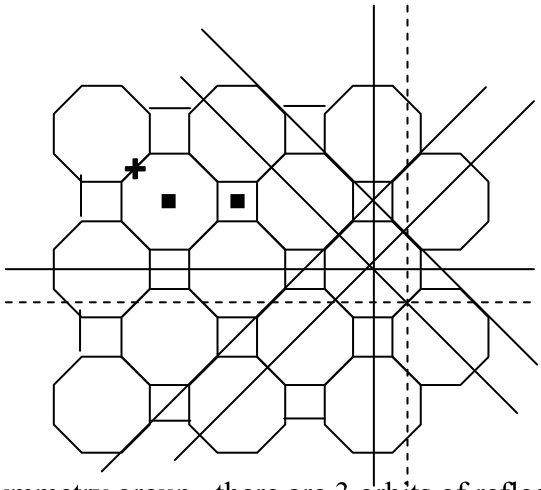
Example for rs: (similar argument for r^3s)



Question 13.

(a) 2 orbits of order 4 rotations, 1 orbit of order 2

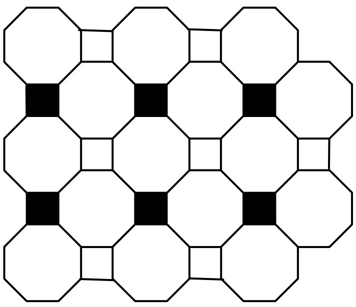
(b)



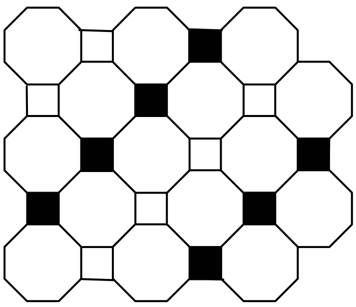
(c) Under the full symmetry group, there are 3 orbits of reflection axes, and one glide orbit.

(d) highest order? 4
reflections? yes
4 directions? yes therefore *p*4*mm*

(e)



(i) *p*4*mm*



(ii) *p*2*mm*

(b) and (c)
NB: under the *translation* group, there are 6 reflection orbits, and 2 glide orbits.

Under the *full* symmetry group, axes that meet at right angles at a rotation centre map to each other by rotation through $\pi/2$.

(d) Algorithm HB p 45–46

(e)
(i) lose 2 diagonal reflections (through the centres of the octagons) and the glides, but still have highest order = 4, and reflections in 4 directions.

(ii) lose the order 4 rotation, and all the vertical & horizontal reflections. [Keep the glides.]

Question 14.

(a) (i) $L(\boldsymbol{a}, \boldsymbol{c})$ is square, since $|\boldsymbol{a}| = |\boldsymbol{c}| = 2$, and $\boldsymbol{a} \cdot \boldsymbol{c} = 0$.
Let $\boldsymbol{a}' = \boldsymbol{a}$, $\boldsymbol{b}' = \boldsymbol{c}$, $\boldsymbol{c}' = \boldsymbol{b}$.
The transition matrix from $(\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}')$ to $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ with integer entries and determinant = –1.

Hence, by Th'm 1.2, $L = L(\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}')$, where $L(\boldsymbol{a}', \boldsymbol{b}')$ is square.

(ii) $\boldsymbol{c}' = (1, 1, 1) = (0, 1, 0) + (-1 + \tfrac{1}{2})\,\boldsymbol{a}' + \tfrac{1}{2}\,\boldsymbol{b}'$,
where (0, 1, 0) is orthogonal to $\boldsymbol{a}'$ and $\boldsymbol{b}'$, so offset = $(\tfrac{1}{2}, \tfrac{1}{2})$.
Vertical separation = $1 = \tfrac{1}{2}|\boldsymbol{a}'|$, so the lattice is **body–centred cubic.**

(b) $|\boldsymbol{d}| = |\boldsymbol{e}| = |\boldsymbol{f}| = \sqrt{50}$, and $\boldsymbol{d} \cdot \boldsymbol{e} = \boldsymbol{e} \cdot \boldsymbol{f} = \boldsymbol{d} \cdot \boldsymbol{f} = 0$
The three vectors are mutually orthogonal, and of equal length, so the lattice is **primitive cubic.**

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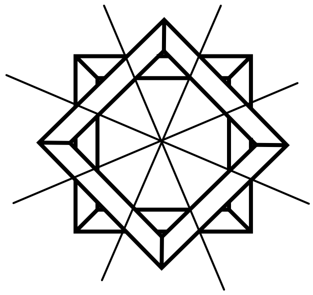
(a) (i) HB p55 Th'm 1.2

(a) (ii) HB p56 Result 3.2

(b) If the numbers look very difficult to manipulate, you should suspect that the lattice is very straightforward. So check lengths and angles.

Question 14 (continued).

(c) (i)



Rotations through π (order 2) about the axes drawn (turning the decoration over).

(ii) $G^{+} \cong D_4$

Central inversion is *not* a symmetry of the decoration

Therefore $G \cong D_8$

(c) (i) Easiest to draw the picture.

(c) (ii) HB p53, Result 5.1

Part II B (Groups)

Question 15

(a) H has n elements.

Then $g^{-1}Hg = \{ g^{-1}hg : h \in H\}$ has **at most** n elements.

Suppose $| g^{-1}Hg | < n$.

Then $g^{-1}h_1g = g^{-1}h_2g$ for some $h_1, h_2 \in H$, where $h_1 \neq h_2$.

Multiplying both sides on the left by g and on the right by g^{-1}, gives

$gg^{-1}h_1g\,g^{-1} = gg^{-1}h_2g\,g^{-1} \Rightarrow h_1 = h_2$, a contradiction.

So $| g^{-1}Hg | = n$.

(b) Closure: Let $g^{-1}h_1g,\ g^{-1}h_2g \in g^{-1}Hg$. Then

$(g^{-1}h_1g)(g^{-1}h_2g) = g^{-1}h_1h_2g \in g^{-1}Hg$

(by cancelling, since H is closed)

Inverse: Let $g^{-1}hg \in g^{-1}Hg$. Then

$(g^{-1}hg)^{-1} = g^{-1}h^{-1}(g^{-1})^{-1} = g^{-1}h^{-1}g \in g^{-1}Hg$ (since $h^{-1} \in H$).

Identity: $e = g^{-1}eg \in g^{-1}Hg$ (since $e \in H$).

(c) X is the set of all n–element subgroups of G.

Closure: For any $H \in X$, $g^{-1}Hg$ contains n elements by part (a), and is a subgroup, by part (b). So $g^{-1}Hg \in X$.

Identity: $e^{-1}he = h$, so $e^{-1}He = H$.

Assoc: If $a, b \in G$, $H \in X$, then

$(ab) \wedge H = (ab)^{-1}H(ab) = b^{-1}a^{-1}Hab = b^{-1}(a \wedge H)b$

$= b \wedge (a \wedge H) \neq$ $\boldsymbol{a \wedge (b \wedge H)}$, ***so associativity does not hold. See comment in next column.***

Note: if we define $g \wedge H = gHg^{-1}$, then we have

$(ab) \wedge H = (ab)H(ab)^{-1} = abH\,b^{-1}a^{-1} = a(b \wedge H)a^{-1} = a \wedge (b \wedge H)$

For the next part (over the page) I'll assume that we do in fact have a group action.

Important note:

There is an error in this question. $g^{-1}Hg$ should be gHg^{-1} throughout!

Everything works for $g^{-1}Hg$, *except* for part (c).

$g \wedge H = g^{-1}Hg$ does not ***in general*** define a group action, since property (c) of Lemma 5.1 (HB p 10) is not satisfied.

However, in the particular case where $G = D_4$ in part (d), it happens to work.

Fortunately, in 1999, very few students did question 15, and the brave souls who tackled it would have been given full marks whether they noticed the error or not. [Most people (including me in my first attempt, and the exam board in their mark scheme), used gHg^{-1} by accident, so didn't notice the error]

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Question 15 (continued)

(c) (continued) Assuming all the group action properties *are* satisfied, $g \wedge H$ defines a homomorphism ϕ: $G \to \Gamma(X)$, where $\Gamma(X)$ is the group of bijections of X, isomorphic to a permutation group.

(d) (i) There are 5 subgroups containing 2 elements in D_4:
$H_1 = \{e, r^2\}$, $H_2 = \{e, s\}$, $H_3 = \{e, rs\}$, $H_4 = \{e, r^2s\}$, $H_5 = \{e, r^3s\}$
Let $X = \{ H_1, H_2, H_3, H_4, H_5\}$
Then the group action defined by $g \wedge H = g^{-1}Hg$ gives the required homomorphism from D_4 to $\Gamma(X) \cong S_5$.

(ii) $r^2 = r[\pi]$ commutes with every element of D_4.
Therefore, for any $x \in H_i$, $i = 1, 2, 3, 4, 5$, we have
$r^{-2}xr^2 = r^{-2}r^2x = x$. So $r^2 \wedge H_i = r^{-2}H_ir^2 = H_i$, and $r^2 \in \text{Ker}(\phi)$.

(c) Definition HB p11

(d) (ii) Remember that $\Gamma(X)$ is a *permutation* group on the symbols H_1, H_2, H_3, H_4, H_5, so the identity of $\Gamma(X)$ is $(H_1)(H_2)(H_3)(H_4)(H_5)$ in cycle notation.

Therefore the kernel of ϕ consists of elements in D_4 that map each element of X to itself under the group action.

Question 16.

(a) $|G| = 2450 = 2 \times 5^2 \times 7^2$

prime	**factors**	**label**
2	2	2a
5	5^2	5a
	5×5	5b
7	7^2	7a
	7×7	7b

There are $1 \times 2 \times 2 = 4$ possible groups.

label	***p*-primary form**	**canonical form**
2a5a7a:	$Z_2 \times Z_{25} \times Z_{49}$	Z_{2450}
2a5b7a:	$Z_2 \times (Z_5 \times Z_5) \times Z_{49}$	$Z_5 \times Z_{490}$
2a5a7b:	$Z_2 \times Z_{25} \times (Z_7 \times Z_7)$	$Z_7 \times Z_{350}$
2a 5b7b:	$Z_2 \times (Z_5 \times Z_5) \times (Z_7 \times Z_7)$	$Z_{35} \times Z_{70}$

(b) $Z_{50} \cong Z_2 \times Z_{25}$ so any group containing a subgroup isomorphic to Z_{50} must also have subgroups isomorphic to Z_2 and Z_{25}. It is clear from the p–primary forms that the only such groups are Z_{2450} and $Z_7 \times Z_{350}$.

Z_{50} contains an element of order 25, and neither of the remaining groups contains such an element. So neither has a subgroup isomorphic to Z_{50}.

(c) Z_{25} has 4 elements of order 5, $Z_5 \times Z_5$ has 24 elements of order 5
Z_{49} has 6 elements of order 7, $Z_7 \times Z_7$ has 48 elements of order 7

so $Z_2 \times Z_{25} \times Z_{49}$ has $4 \times 6 = 24$ elements of order 35

$Z_2 \times (Z_5 \times Z_5) \times Z_{49}$ has $24 \times 6 = 144$ elements of order 35

$Z_2 \times Z_{25} \times (Z_7 \times Z_7)$ has $4 \times 48 = 192$ elements of order 35

$Z_2 \times (Z_5 \times Z_5) \times (Z_7 \times Z_7)$ has $24 \times 48 = 1152$ elements of order 35

(a) Canonical decomposition HB p33 Theorem 3.2
p–primary decomposition HB p38

(c) Result 2.6 HB p22
$a \in G$, $b \in H$, $(a,b) \in G \times H$, then $|(a,b)| = \text{lcm}(|a|, |b|)$.

$35 = 5 \times 7$.
Write $G = A \times B \times C$, where A, B and C are the 2–, 5– and 7–primary components respectively. Then for $(a,b,c) \in A \times B \times C$, if $|(a,b,c)| = 35$, then $|a| = 1$, $|b| = 5$ and $|c| = 7$.

Then count the number of element of order 5 in B, and those of order 7 in C.

Other results: For prime p,

a group of order p ($\cong Z_p$) has $p - 1$ elements of order p

Z_{p^2} has exactly 1 subgroup of order p, so has $p - 1$ elements of order p
(Th'm 4.2 (b) HB p25)

$Z_p \times Z_p$ has no element of order p^2, so the $p^2 – 1$ non–identity elements all have order p (possible orders are 1, p and p^2 by Lagrange's theorem).

Question 17.

$|G| = 3^2 \times 13$. Let n_p be the number of Sylow p-subgroups of G.

(a) $n_{13} \equiv 1 \pmod{13}$ and $n_{13} \mid 9$, so $n_{13} = 1$.

Thus the Sylow 13–subgroup, K, is unique, hence normal in G (conjugate only to itself), and has order 13.

(b) $n_3 \equiv 1 \pmod 3$ and $n_3 \mid 13$, so $n_3 = 1$ or 13.

(c) The quotient group G/K has order 117/13 = 9, the square of a prime.

Therefore $G/K \cong Z_9$ or $Z_3 \times Z_3$, both Abelian groups. In either case, G/K has a normal subgroup, say $\overline{H}$, of order 3.

By the Correspondence Theorem, G has a normal subgroup, H, containing K, of order $|K| \times |\overline{H}| = 13 \times 3 = 39$.

(d) If G has a normal subgroup N of order 9, then Theorem 4.1 (HB p49) applies:

i) $|G| = |K| \times |N|$ $(117 = 13 \times 9)$

ii) $|K| = 13$, $|N| = 9$, 13 and 9 are coprime

iii) K and N are normal in G.

Therefore $G \cong K \times N$.

$|K| = 13$, a prime, so $K \cong Z_{13}$

$|N| = 9$, the square of a prime, so $N \cong Z_9$ or $N \cong Z_3 \times Z_3$.

Therefore either $G \cong Z_{13} \times Z_9 \cong Z_{117}$

or $G \cong Z_{13} \times Z_3 \times Z_3 \cong Z_3 \times Z_{39}$.

(a) & (b) Theorem 3.1 (Summary of the Sylow results) HB p48

(c) Correspondence Th'm Th'm 5.2 HB p37

$K \le H \le G$, K normal in G, so K normal in H.

Then $H/K \le G/K$, and H/K is normal in G/K *if and only if* H is normal in G.

$\overline{H} = H/K$ is the set of cosets of K in H.

There are $|K|$ element of G in each coset, and $|\overline{H}|$ cosets.

The cosets partition H, so $|H| = |\overline{H}| \times |K|$.