1996 Brief solutions to exam

1 (i) Let Y be no. of questions per call:

$Y \sim G_1(p)$ so $E(Y) = \frac{1}{p} = 2.5$

so $p = 0.4$

and $\Pi_Y(s) = \frac{0.4s}{1-0.6s}$ or $\frac{2s}{5-3s}$.

$$P(Y \geq 2) = 1 - P(Y \leq 2)$$
$$= 1 - (p_1 + p_2)$$
$$= 1 - (p + pq)$$
$$= 1 - (0.4 + 0.24) = \underline{0.36}$$

(ii) No. of calls in t hrs $\sim$ Poisson(λt), $\lambda = 4$.

No of calls in $\frac{1}{2}$ hr, $N \sim$ Poisson (2).

$\Pi_N(s) = e^{-2(1-s)}$.

Total no. of questions in $\frac{1}{2}$ hr,

$$Z = Y_1 + \cdots + Y_N$$
$$\Pi_Z(s) = \Pi_N(\Pi_Y(s)) = e^{-2\left(1 - \frac{2s}{5-3s}\right)}$$
$$= e^{-10(1-s)/(5-3s)}$$

(iii) Expected no. of questions in $\frac{1}{2}$ hour is

$$\mu \lambda t = 2.5 \times 4 \times \tfrac{1}{2} = 5$$

($\underline{\text{or}}$ $\Pi_Z'(1) = 5$ $\underline{\text{or}}$ $E(Z) = E(N)E(Y) = 2 \times 2.5 = 5$)

2 (i) $p_{JS}\,p_{SB} = \frac{1}{2} \times \frac{2}{3} = \frac{2}{9}$

(ii) $$(\pi_B\ \pi_J\ \pi_S) = (\pi_B\ \pi_J\ \pi_S)\begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$$ so

$$\pi_B = \tfrac{2}{3}\pi_J + \tfrac{2}{3}\pi_S$$
$$\pi_J = \tfrac{2}{3}\pi_B$$
$$\pi_S = \tfrac{1}{3}\pi_B + \tfrac{1}{3}\pi_J + \tfrac{1}{3}\pi_S \quad \text{and} \quad \pi_B + \pi_J + \pi_S = 1.$$

This gives $\pi_B = \frac{2}{5}$, $\pi_J = \frac{4}{15}$, $\pi_S = \frac{1}{3}$.

She reads a book on 2/5 of weekends, does a jigsaw 4/15 of weekends and sewing on $\frac{1}{3}$ of weekends.

(iii) Expected time till she next reads a book is

$$1/\pi_B = 2\tfrac{1}{2} \text{ weeks.}$$

3 (i) M/M/3

$\lambda = 30$ per hour or $\frac{1}{2}$ per minute $\qquad \varepsilon = 15$ per hour or $\frac{1}{4}$ per minute

$$\rho = \frac{\lambda}{n\varepsilon} = \frac{30}{3 \times 15} = \frac{2}{3}.$$

(ii) Probability distribution of X, the equilibrium queue size, is given by

$$p_x = \begin{cases} \frac{1}{K} \frac{3^x}{x!} \rho^x & x = 0, 1, 2 \\ \frac{1}{K} \frac{3^3}{3!} \rho^x & x = 3, 4, 5, \ldots \end{cases}$$

where

$$K = 1 + 3\rho + \frac{(3\rho)^2}{2!} + \frac{(3\rho)^3}{3!(1-\rho)}$$
$$= 1 + 2 + 2 + 4$$
$$= 9$$

(a) $p_0 = \frac{1}{K} = \frac{1}{9}$

(b) $p_1 = \frac{2}{9}$, $p_2 = \frac{2}{9}$ so $p_{3 \text{ or more}} = \frac{4}{9}$.

Mean number of cashiers busy is

$$0 \times \tfrac{1}{9} + 1 \times \tfrac{2}{9} + 2 \times \tfrac{2}{9} + 3 \times \tfrac{4}{9}$$
$$= 2$$

4 (i) $\beta = 0.75$, $\gamma = 0.4$, $n+1 = 31$ so

$$\rho = \frac{n\gamma}{\beta} = \frac{30 \times 0.4}{0.75} = 16$$

(ii) $$y_{max} = x_0 + y_0 - \rho - \rho \log \frac{x_0}{\rho}$$
$$= 31 - 16 - 16 \log \frac{25}{16} \simeq \underline{7.86}$$

(iii) x_∞ is smaller solution of

$$x = x_0 \exp\left(\frac{x - (x_0 + y_0)}{\rho}\right)$$

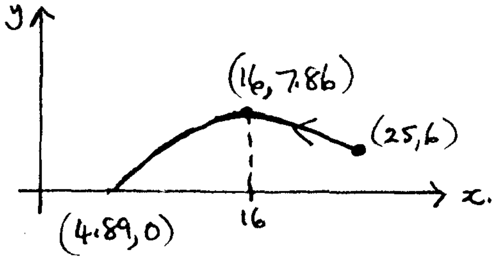
i.e. $$x = 25 \exp\left(\frac{x - 31}{16}\right)$$

Using formula iteration with $x_{\infty,0} = 0$ gives $x_{\infty,1} = 3.6016$, $x_{\infty,2} = 4.5108$, 4.7746, 4.8539, 4.8780, 4.8854, 4.8876, 4.8883, $x_{\infty,9} = 4.8886$, $x_{\infty,10} = 4.8886$

So $x_\infty \simeq \underline{4.89}$.

The number who have had the cold and recovered is $31 - 4.89 = \underline{26.11}$.

(iv)



5 (i) $F(s) = 0.5s^2 + 0.5s^3$ so $U(s) = \frac{1}{1 - (0.5s^2 + 0.5s^3)}$

This expands to give

$$U(s) = 1 + (0.5s^2 + 0.5s^3) + (0.5s^2 + 0.5s^3)^2 + (0.5s^2 + 0.5s^3)^3 + \cdots$$

Hence $u_4 = 0.5^2 = 0.25$

$u_5 = 2 \times 0.5 \times 0.5 = 0.5$

$u_6 = 0.5^2 + 0.5^3 = 0.375$.

5 ctd. (ii) W_5 has p.g.f.

$$F(s)^5 = (0.5s^2+0.5s^3)^5$$
$$= s^{10}(0.5+0.5s)^5$$

so $W_5 = 10 + K_5$ where $K_5 \sim B(5, 0.5)$.

(a) Range of W_5 is $\{10, 11, 12, 13, 14, 15\}$.

(b) $P(W_5 = 13) = P(K_5 = 3) = \binom{5}{3} 0.5^5$

$$= 10 \times \tfrac{1}{32} = \underline{0.3125}$$

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6 (i) Require $P(H(\tfrac{1}{2}) \geq \tfrac{30}{25}) = P(H(\tfrac{1}{2}) \geq 1.2)$.

$X(t) \sim N(0, 0.08t)$ so $X(\tfrac{1}{2}) \sim N(0, 0.04)$

$$P(H(\tfrac{1}{2}) \geq 1.2) = P(X(\tfrac{1}{2}) \geq \log 1.2)$$
$$= 1 - \Phi\left(\frac{\log 1.2}{0.2}\right)$$
$$= 1 - \Phi(0.912)$$
$$= 1 - 0.8191 \simeq \underline{\underline{0.181}}$$

(ii) Require

$$P(H(2) < 0.8 \mid H(0) = H(3) = 1)$$
$$= P(X(2) < \log 0.8 \mid X(0) = X(3) = 0)$$

Brownian bridge: the conditional distribution of $X(2)$ has mean 0 and variance

$$\frac{(t_2-t)(t-t_1)}{t_2-t_1}\sigma^2 = \frac{1\times 2}{3} \times 0.08 = \frac{0.16}{3}$$

So $P(X(2) < 0.8 \mid X(0) = X(3) = 0)$

$$= \Phi\left(\frac{\log 0.8}{\sqrt{0.16/3}}\right) = \Phi(-0.966) = \underline{\underline{0.1670}}$$

7(b) (i) The object-to-nearest-object distance S has a Rayleigh distribution with parameter β given by

$$\beta^2 = 2\pi\lambda = 8\pi \times 10^{-5}.$$

So $E(S) = \frac{1}{\beta}\sqrt{\frac{\pi}{2}} = 25\sqrt{10} \simeq \underline{79.1\,m}.$

$$P(S < 100) = F(100) = 1 - e^{-\pi\lambda 100^2}$$
$$= 1 - e^{-0.4\pi}$$
$$= \underline{0.7154}$$

(ii) Number of bushes in region

$N \sim \text{Poisson}(400 \times 250 \times 4 \times 10^{-5})$

i.e. Poisson(4)

$$P(\text{at least 1 bush}) = 1 - p_0$$
$$= 1 - e^{-4} = \underline{0.9817}$$

P(not more than 2 bushes)

$$= p_0 + p_1 + p_2$$
$$= e^{-4} + 4e^{-4} + \frac{4^2 e^{-4}}{2!} \quad \text{(or use Neave)}$$
$$= 13e^{-4} = \underline{0.2381}.$$

(iii) Poisson(4) has prob. function & c.d.f.:

n	0	1	2	3
$p(n)$	0.0183	0.0733	0.1465	0.1954
$F(n)$	0.0183	0.0916	0.2381	0.4335

so $F(2) < u \leq F(3)$

and simulated no. of bushes is $\underline{\underline{3}}$.

(iv) For each bush, two independent random numbers from $U(0,1)$ are taken (u_1, u_2 say). The coordinates of the bush, are then

$$(x, y) = (400u_1, 250u_2).$$

8(b)(i) $E(Z_i) = 4p - q$ or $5p - 1$

$E(Z_i^2) = 16p + q$ or $15p + 1$

$V(Z_i) = 16p + q - (4p-q)^2 = 25pq$

So $E(X_n) = n(4p - q)$

$V(X_n) = 25npq$.

(ii) If there are a steps of $+4$ and $(n-a)$ steps of -1 in the first n steps, then

$$X_n = 4a - (n-a) = 5a - n.$$

W_n, the number of steps of $+4$ in the first n steps, has a binomial distribution, $B(n,p)$, so

$$P(W_n = a) = \binom{n}{a} p^a q^{n-a}, \quad a = 0, 1, \ldots, n.$$

Hence

$$P(X_n = 5a - n) = \binom{n}{a} p^a q^{n-a}, \quad a = 0, 1, \ldots, n.$$

Let $x = 5a - n$, so $a = (n+x)/5$ and $n - a = (4n-x)/5$, and so

$$P(X_n = x) = \binom{n}{(n+x)/5} p^{\frac{n+x}{5}} q^{\frac{4n-x}{5}}.$$

The range of X_n is

$$\{-n, -n+5, -n+10, \ldots, 4n-10, 4n-5, 4n\}.$$

(iii)(a) $P(X_7 = -3) = 0$ (b) $P(X_6 = 4) = 15p^2q^4$

(iv) Return to origin is possible only after $5, 10, 15, \ldots$ steps, so

$$f_5 = u_5 = P(X_5 = 0) = 5pq^4.$$

(v) $P(X_{10} > 0) = 1 - P(X_{10} \leq 0)$

$$= 1 - \{P(X_{10} = -10) + P(X_{10} = -5) + P(X_{10} = 0)\}$$
$$= 1 - \{(\tfrac{1}{2})^{10} + 10 \times (\tfrac{1}{2})^{10} + 45 \times (\tfrac{1}{2})^{10}\}$$
$$= 1 - \frac{56}{1024} = 1 - \frac{7}{128}$$
$$= \frac{121}{128} \text{ or } 0.9453.$$

IT16 old

9/ (i) $\nu(1-s)\frac{\partial\Pi}{\partial s} - \frac{\partial\Pi}{\partial t} = \lambda(1-s)\Pi$

so $f = \nu(1-s)$ $\quad g = -1$ $\quad h = \lambda(1-s)\Pi$

(ii) $\frac{ds}{\nu(1-s)} = \frac{dt}{-1} = \frac{d\Pi}{\lambda(1-s)\Pi}$

$\int \frac{ds}{\nu(1-s)} = -\int dt$

$-\log(1-s) = -\nu t + \text{const}$

$(1-s) = e^{\nu t} \times \text{const}$

$\underline{C_1 = (1-s)e^{-\nu t}}$

$\int \frac{ds}{\nu} = \int \frac{d\Pi}{\lambda\Pi}$

$\frac{s}{\nu} = \frac{1}{\lambda}\log\Pi + \text{const}$

$\Pi = e^{\lambda s/\nu} \times \text{const}$

$\underline{C_2 = \Pi e^{-\lambda s/\nu}}$

(iii) Using $C_2 = \psi(C_1)$, $\quad \Pi(s,t) = e^{\lambda s/\nu}\psi\left((1-s)e^{-\nu t}\right)$.

(iv) $\Pi(s,0) = 1$, so $e^{\lambda s/\nu}\psi(1-s) = 1$

so $\psi(1-s) = e^{-\lambda s/\nu}$

Let $x = 1-s$, so $s = 1-x$ & $\psi(x) = e^{-\lambda(1-x)/\nu}$

and $\Pi(s,t) = e^{\lambda s/\nu}\exp\left\{\frac{-\lambda}{\nu}\left(1-(1-s)e^{-\nu t}\right)\right\}$

$= \exp\left[\frac{-\lambda}{\nu}(1-e^{-\nu t})(1-s)\right]$.

Hence $X(t) \sim \text{Poisson}\left(\frac{\lambda}{\nu}(1-e^{-\nu t})\right)$.

(v) After a long time $X \approx \text{Poisson}\left(\frac{\lambda}{\nu}\right)$.

Working in hours $\lambda = 360$, $\frac{1}{\nu} = \frac{1}{3}$ so $\nu = 3$, so

$X \sim \text{Poisson}(120)$

which has mean 120.

(vi) For each swimmer, prob. of leaving within 1 hour is $1 - e^{-\nu} = 1 - e^{-3} = 0.9502129316\ldots$

P(all 50 swimmers leave within 1 hour)

$= (0.9502129316\ldots)^{50} = 0.07781$.

P(exactly 49 leave) is $50e^{-3}(1-e^{-3})^{49} = 0.20385$

So the probability that at least two swimmers remain after an hour is

$1 - 0.07781 - 0.20385 = 0.71834 \simeq \underline{0.718}$.

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10 (b)(i) $H(x) = \int_0^x \frac{1}{90-u}\,du = \left[-\log(90-u)\right]_0^x$

$= -\log\frac{90-x}{90}$

So $Q(x) = e^{-H(x)} = \frac{90-x}{90} = 1 - \frac{x}{90}$, $0 \le x \le 90$

(ii) Lifetime is $U(0,90)$ so mean lifetime is $\underline{45 \text{ years}}$

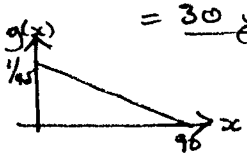
(OR $e_0 = \int_0^\infty Q(x)\,dx = \int_0^{90} 1 - \frac{x}{90}\,dx = 45$ years).

(iii) $g(x) = \frac{Q(x)}{e_0} = \frac{1}{45}\left(1 - \frac{x}{90}\right)$, $\quad 0 \le x < 90$.

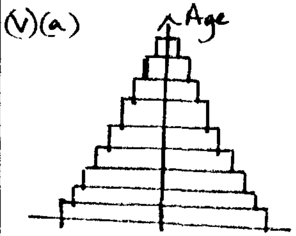
Mean age $= \int_0^{90} x\,g(x)\,dx = \frac{1}{45}\int_0^{90} x - \frac{x^2}{90}\,dx$

$= \underline{30 \text{ years}}$.

(iv)

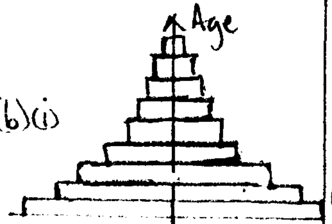


(v)(a)

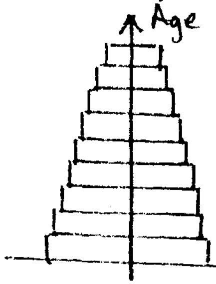


(b)(ii) Stable & declining:
A smaller proportion of the population is in the younger age groups, so the base is narrower and the top is wider

(v)(b)(i)



Stable and growing;
A larger proportion of the population is in the younger age groups than for the stationary population, so the base of the pyramid is wider & the top narrower



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11 (i) P(green-eyed) = P(bb) = $q^2 = 0.64$

so $q = 0.8$

P('pure' blue-eyed) = P(BB) = $p^2 = 0.2^2 = \underline{0.04}$

(ii) $P(BB \mid \text{blue}) = \frac{P(BB)}{P(\text{blue})} = \frac{0.04}{0.36} = \frac{1}{9}$

(iii) Initially we have

BB	Bb	bb
$\frac{1}{9}$	$\frac{8}{9}$	0
(u	2v	w)

Hardy-Weinberg Theorem says that at the next generation and in subsequent generations the gene frequencies stabilise.

$P(BB) = (u+v)^2 = \frac{25}{81}$

$P(Bb) = 2(u+v)(v+w) = \frac{40}{81}$

$P(bb) = (v+w)^2 = \frac{16}{81}$

$\frac{65}{81}$ of animals will have blue eyes and $16/81$ will have green eyes.

(iv) Require

P(both parents BB | offspring blue-eyed)

$= \frac{\text{P(offspring blue-eyed | parents BB) P(parents BB)}}{\text{P(offspring blue-eyed)}}$ (Bayes' formula)

$= \frac{1 \times \left(\frac{25}{81}\right)^2}{\frac{65}{81}} = \frac{625}{5265}$ or $\frac{125}{1053}$

$\underline{\underline{\simeq 0.1187}}$

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12 On old Unit 15 – Ignore.