1997 EXAM

KP'S ANSWERS / GUESSES! ~ CAVEAT EMPTOR!

1. HOPKINS' TEST.

$\sum R^2 = 1189$ $\quad H = \frac{1189}{426} = 2.791$

$\sum S^2 = 426$

$F_{20}^{20}(0.05) = 0.471$ $\quad F_{20}^{20}(0.95) = 2.124$

$F_{20}^{20}(0.025) = 0.406$ $\quad F_{20}^{20}(0.975) = 2.464$

H IS SIG AT $\alpha = 0.05$

EVIDENCE OF CLUSTERING

2. (i)(a) $\pi(0) = \left(\frac{2}{5}\right)^2 = 0.16$ = PROB OF EXTINCTION BY 1ST GENERATION

$\pi^2(0) = \left(\frac{2}{5-3\times 0.16}\right)^2 =$ ——— 2ND ———

$\simeq \underline{0.1958}$

(b) $\pi^3(0) \doteq 0.2054$

$\therefore$ PROB OF EXTINCTION AT 3RD GEN $= \pi^3(0) - \pi^2(0)$

$\simeq \underline{0.0096}$

(ii) NEED TO SOLVE $g = \left(\frac{2}{5-3g}\right)^2$ (SMALLEST +VE SOL^N REQUIRED)

USING FORMULA ITERATION WITH $g_{n+1} = \left(\frac{2}{5-3g_n}\right)^2$ AND $g_0 = 0.2$

GIVES $g_1 = 0.206612$ $\quad g_2 = 0.208487$ $\quad g_3 = 0.209024$

$g_4 = 0.209178$ $\quad g_5 = 0.209222$ $\quad g_6 = 0.209234$

$g_7 = 0.209238$ ...

SO TO 4 D.P. PROB. OF EVENTUAL EXTINCTION IS $\underline{0.2092}$

3. (i) GAMBLER'S RUIN $p = 0.4$ $a = 6$ $j = 3$

$$q_3 = \text{PROB BROTHER WINS (JAMES IS RUINED)} = \frac{\left(\frac{0.6}{0.4}\right)^3 - \left(\frac{0.6}{0.4}\right)^6}{1 - \left(\frac{0.6}{0.4}\right)^6} \simeq \underline{0.771}$$

(ii) D_3 = EXPECTED DURATION (AV NO OF GAMES PER CONTEST) $= \dfrac{3}{0.2} - \dfrac{6}{0.2}\left(\dfrac{1-\left(\frac{0.6}{0.4}\right)^3}{1-\left(\frac{0.6}{0.4}\right)^6}\right)$

$= \underline{8\tfrac{1}{7} \text{ GAMES}}$

(iii) $D_3^* = 3(6-3) = \underline{9 \text{ GAMES}}$

(iv) REQUIRE $q_3 = \left(\dfrac{0.2}{0.8}\right)^3 = \underline{0.015625}$

4. (a)(i) $M/G/2$ (ii) $D/M/1$

(b)(i) $\rho = \dfrac{18}{2 \times 15} = 0.6$

(ii) $K = 1 + 2 \times 0.6 + \dfrac{(2 \times 0.6)^2}{2(1-0.6)} = 3.1$

$p_0 = \dfrac{1}{3.1}\,\dfrac{2^0}{0!}\,\rho^0 = \dfrac{1}{3.1} \simeq \underline{0.323}$

(iii) $p_1 = \dfrac{1}{3.1}\,\dfrac{2^1}{1!}\,\rho^1 = \dfrac{2 \times 0.6}{3.1} \simeq 0.387$

$\therefore$ PROB (IMMEDIATE ATT^{N}) $= p_0 + p_1 \simeq \underline{\underline{0.710}}$

(iv) REQ PROB $\left(M(30) > \frac{3}{60}\right) = e^{-30 \times \frac{3}{60}} = e^{-1.5} \simeq \underline{\underline{0.223}}$

5 (i) $Q(x) = \int_x^3 \frac{2u}{9}\,du = \left[\frac{u^2}{9}\right]_x^3 = 1 - \frac{x^2}{9} \quad 0 \le x \le 3$

(ii) $e_0 = \int_0^\infty Q(x)\,dx = \int_0^3 \left(1 - \frac{x^2}{9}\right)dx$

$= \left[x - \frac{x^3}{27}\right]_0^3 = 3 - 1 = \underline{2 \text{ WEEKS}}$

(iii) STATIONARY, so $g(x) = Q(x)/e_0$

$\eta = \int_0^\infty x\,g(x)\,dx = \frac{1}{e_0}\int_0^3 x\left(1 - \frac{x^2}{9}\right)dx$

$= \frac{1}{2}\left[\frac{x^2}{2} - \frac{x^4}{36}\right]_0^3 = \frac{9}{8} = \underline{1\tfrac{1}{8} \text{ WEEKS}}$

(iv) $\int_0^1 g(x)\,dx = \frac{1}{2}\int_0^1 Q(x)\,dx = \frac{1}{2}\left[x - \frac{x^3}{27}\right]_0^1 = \underline{\frac{13}{27} \text{ WKS}}$

(v) $1 - P(X > 1) = 1 - Q(1) = \underline{\frac{1}{9} \text{ WKS}}$

6. (i) $H(t) = \int_0^t h(v)\,dv = \left[\frac{v^2}{32}\right]_0^t = \frac{t^2}{32}$

$Q(t) = 1 - G(t) = e^{-H(t)} = \underline{e^{-t^2/32}}$

$g(t) = \frac{t}{16} e^{-t^2/32}$ $\quad$ $\underline{\text{RAYLEIGH WITH } \beta = \tfrac{1}{4}}$

(ii) MEAN (RAYLEIGH) $= \frac{1}{\beta}\sqrt{\frac{\pi}{2}} \doteq \underline{5.013 \text{ YRS}}$

FOR MEDIAN SOLVE $Q(t) = \frac{1}{2}$ i.e. $e^{-t^2/32} = \frac{1}{2}$

$t = \sqrt{32 \log 2} \doteq \underline{4.710 \text{ YRS}}$

(iii) $Q(10.026) \doteq 0.0432$

(iv) $Q(z)\,Q(t) = e^{-z^2/32}\,e^{-t^2/32} = e^{-(z^2+t^2)/32}$

$Q(z+t) = e^{-(z+t)^2/32}$

NOW $z > 0$ & $t > 0$ so $(z+t)^2 > z^2 + t^2$ so $e^{-(z+t)^2/32} < e^{-(z^2+t^2)/32}$

SO $Q(z)\,Q(t) > Q(z+t)$ SO $\underline{\underline{\text{NBU}}}$

7.(a) (i) THE EMISSION OF ALPHA-PARTICLES FROM A SMALL AMOUNT OF RADIOACTIVE MATERIAL

(ii) CUSTOMERS ARRIVE AT AN M&S CHECKOUT (CLOTHES) ~ ACCORDING TO A POISSON PROCESS
THE NUMBER OF ITEMS TO BE PAID
FOR BY A CUSTOMER IS A RANDOM VARIABLE.
THIS IS A COMPOUND POISSON PROCESS, THE VARIABLE BEING
THE NUMBER OF ITEMS SOLD IN TOTAL IN A GIVEN INTERVAL OF TIME

(b) (i) $P(0) = e^{-45/12} \simeq \underline{0.024}$

(ii) $M(1.8)$ — MEAN $= \frac{1}{1.8}$ HRS $= \underline{33\frac{1}{3}\text{ MINS}}$

(iii)! $P(1.8)$ $\quad P(\geq 3) = 1 - (P(0) + P(1) + P(2))$

$\simeq \underline{0.269}$

(iv) $P(3\text{ LORRIES}) = \dfrac{1.8^3 e^{-1.8}}{3!} \simeq 0.161$

$P(7\text{ NON-LORRIES}) = \dfrac{13.2^7 e^{-13.2}}{7!} \simeq 0.026$

$\therefore$ P. REQUIRED $\simeq 0.161 \times 0.026 \simeq \underline{0.00412}$ (CALC ACCURACY)

(v) REQ $P(X=3)$ WHERE $X \sim B(10, 0.12)$

i.e. $\binom{10}{3} 0.12^3\, 0.88^7 \simeq \underline{0.085}$

8 (i) HISTORY IRRELEVANT

TRANSITION PROBABILITIES CONSTANT

FIRST ASSUMPTION MIGHT NOT HOLD IN A PERIOD OF SOCIAL OR ECONOMIC UPHEAVAL (e.g. PIT CLOSURES IN A COALFIELD REGION)

SECOND ASSUMPTION IS GENERALLY BELIEVED TO HOLD IN THE SHORT TERM, THOUGH TRANSITION PROBABILITIES HAVE BEEN SEEN TO CHANGE IN THE LONG TERM AS THE NATURE OF SOCIETY HAS CHANGED.

(ii) (a) REQUIRE $p_{12} \times p_{22} = 0.5 \times 0.6 = 0.3$

(b) REQUIRE $(p_{11} \times p_{13}) + (p_{12} \times p_{23}) + (p_{13} \times p_{33})$

$= (0.4 \times 0.1) + (0.5 \times 0.3) + (0.1 \times 0.6) = 0.25$

(iii) LET THE LONG-RUN PROBABILITIES BE a, b, c $(a+b+c=1)$.

THEN $[a \ b \ c]\left[\ P\ \right] = [a \ b \ c]$

GIVES
$$\left.\begin{aligned} a + b + c &= 1 \\ -0.6a + 0.1b + 0.1c &= 0 \\ 0.5a - 0.4b + 0.3c &= 0 \end{aligned}\right\} \rightarrow \quad a = \frac{7}{49},\quad b = \frac{23}{49},\quad c = \frac{19}{49}$$

(iv) $[0.1 \ \ 0.3 \ \ 0.6]\,P = [0.13 \ \ 0.41 \ \ 0.46]$

$[0.1 \ \ 0.3 \ \ 0.6]\,P^2 = [0.13 \ \ 0.41 \ \ 0.46]\,P = [0.139 \ \ 0.449 \ \ 0.412]$

9 (a) 2 SEXES; MULTIPLE BIRTHS; GAPS BETWEEN SUCCESSIVE BIRTHS; BIRTH RATE DEPENDS ON AGE; ...

(b) (i) $\beta(1-s)(2-s)\dfrac{\partial \pi}{\partial s} - \dfrac{\partial \pi}{\partial t} = 0$

$f(s,t,\pi) = \underline{\beta(1-s)(2-s)}$; $g(s,t,\pi) = \underline{-1}$; $h(s,t,\pi) = \underline{0}$

(ii) $\dfrac{ds}{\beta(1-s)(2-s)} = \dfrac{dt}{-1} = \dfrac{d\pi}{0}$

USING 1ST PAIR $\displaystyle\int \frac{ds}{(1-s)(2-s)} = -\beta \int dt + \text{CONST}$

i.e (USING PARTIAL FRACTIONS) $-\log(1-s) + \log(2-s) = -\beta t + \text{CONST}$

i.e. $\log\left(\dfrac{2-s}{1-s}\right) = -\beta t + \text{CONST}$

i.e. $\dfrac{2-s}{1-s} = C_2 e^{-\beta t}$ (WHERE $C_2 = e^{\text{CONST}}$)

i.e. $\underline{C_2 = \dfrac{2-s}{1-s} e^{\beta t}}$

(iii) GENERAL SOL^{n} IS $\pi(s,t) = \psi\left(\frac{2-s}{1-s} e^{\beta t}\right)$

(iv) GIVEN $\pi(s,0) = s^4$ SO $s^4 = \psi\left(\frac{2-s}{1-s}\right)$

LETTING $x = \dfrac{2-s}{1-s}$ SO THAT $s = \dfrac{2-x}{1-s}$ AND $\psi(x) = \left(\dfrac{2-x}{1-x}\right)^4$

THUS $\underline{\pi(s,t) = \left(\dfrac{2 - \frac{2-s}{1-s}e^{\beta t}}{1 - \frac{2-s}{1-s}e^{\beta t}}\right)^4}$

(v) $P(X_{n+1} = x+1 \mid X_n = x) = P(\text{BIRTH}) = \dfrac{\beta}{\beta + 2\beta} = \dfrac{1}{3}$

$P(\text{DEATH}) = \dfrac{2}{3}$

$\therefore$ FROM GAMBLER'S RUIN EXTINCTION IS CERTAIN SINCE $p < q$

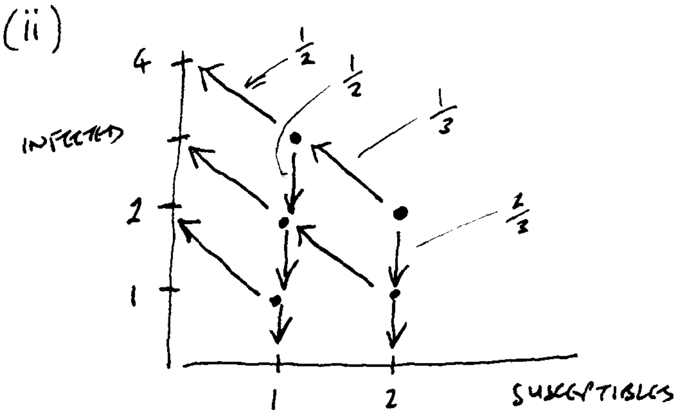
(vi) GAMBLER'S RUIN WITH $j = 4$ $a = 7$ $p = \frac{1}{3}$

PROB $= \dfrac{1-2^4}{1-2^7} \simeq \underline{0.118}$

10. (a) IN THE SIMPLE EPIDEMIC MODEL INDIVIDUALS NEVER RECOVER AND THE WHOLE POPULATION EVENTUALLY BECOMES INFECTED. IN THE GENERAL MODEL SUFFERERS "RECOVER", AND THE EPIDEMIC MAY DIE OUT BEFORE ALL ARE INFECTED.

(b) (i) $\rho = \dfrac{n\gamma}{\beta} = \dfrac{(4-1)\times 2}{6} = 1$

(ii)



LET X BE THE NUMBER OF "SURVIVORS"

$P(X=2) = \left(\frac{2}{3}\right)^2 = \underline{\underline{\frac{4}{9}}}$

$P(X=1) = \frac{1}{3}\times\left(\frac{1}{2}\right)^3 + \frac{2}{3}\times\frac{1}{3}\times\left(\frac{1}{2}\right)^2$

$= \underline{\underline{\frac{7}{72}}}$

$P(X=0) = \left(\frac{1}{3}\times\frac{1}{2}\right) + \left(\frac{1}{3}\times\left(\frac{1}{2}\right)^2\right) + \left(\frac{1}{3}\times\left(\frac{1}{2}\right)^3\right) + \left(\frac{2}{3}\times\frac{1}{3}\times\frac{1}{2}\right) + \left(\frac{2}{3}\times\frac{1}{3}\times\left(\frac{1}{2}\right)^2\right) = \underline{\underline{\frac{33}{72}}}$

(OR BY SUBTRACTION)

(iii) $\rho = \dfrac{(29+2-1)\times 2}{5} = 12$

(iv) $P(\text{MAJOR OUTBREAK}) = 1 - \left(\frac{12}{29}\right)^2 \simeq \underline{0.829}$

(v) NEED TO SOLVE $12\log(29/x) - (29-x) = 2$

USE $x_{n+1} = 29\exp\left(\dfrac{x_n - 31}{12}\right)$ WITH $x_0 = 5$ (SAY)

$x_1 = 3.322$; $x_2 = 2.889$; $x_3 = 2.786$; $x_4 = 2.763$;

$x_5 = 2.757$; $x_6 = 2.756$; $x_7 = 2.756$

SO $\underline{\underline{2.76}}$ TO 2 D.P.

1. (i) (a) $P(AA) = 0.8^2 = \underline{\underline{0.64}}$

(b) $P(Aa) = 2 \times 0.8 \times 0.2 = \underline{\underline{0.32}}$

(c) $P(aa) = 0.2^2 = \underline{\underline{0.04}}$

(ii) $P(AA \mid \sim aa) = \dfrac{0.64}{0.64 + 0.32} = \underline{\underline{\dfrac{2}{3}}}$

$P(Aa \mid \sim aa) = \underline{\underline{\dfrac{1}{3}}}$

(iii)

PARENTS	PROB	CHILD AA	CHILD Aa	CHILD aa
AA × AA	$4/9$	1	0	0
AA × Aa	$4/9$	$\frac{1}{2}$	$\frac{1}{2}$	0
Aa × Aa	$1/9$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(a) $P(\text{CHILD } AA) = \dfrac{4}{9} + \left(\dfrac{1}{2} \times \dfrac{4}{9}\right) + \left(\dfrac{1}{4} \times \dfrac{1}{9}\right) = \dfrac{25}{36}$

(b) $P(\text{CHILD } Aa) = \left(\dfrac{1}{2} \times \dfrac{4}{9}\right) + \left(\dfrac{1}{2} \times \dfrac{1}{9}\right) = \dfrac{10}{36}$

(c) $P(\text{CHILD } aa) = \dfrac{1}{4} \times \dfrac{1}{9} = \dfrac{1}{36}$

(iv) $P\left(\begin{matrix}\text{BOTH PARENTS} \\ Aa\end{matrix} \middle| \begin{matrix}\text{CHILD} \\ Aa\end{matrix}\right) = \dfrac{P\left(\text{CHILD } Aa \middle| \begin{matrix}\text{BOTH PARENTS} \\ Aa\end{matrix}\right) P\left(\begin{matrix}\text{BOTH PARENTS} \\ Aa\end{matrix}\right)}{P(\text{CHILD } Aa)}$

$$= \frac{\frac{1}{2} \times \frac{1}{9}}{\frac{10}{36}} = \underline{\underline{\frac{1}{5}}}$$

12 (a) APPROPRIATE WHERE WE KNOW THE INITIAL & FINAL STATES OF A BROWNIAN MOTION, AND WISH TO KNOW THE PROBABILITY DISTRIBUTION OF AN INTERMEDIATE STATE.

(b) (i) $X(2) \sim N(0, 1600)$

$$P(|X(2)| > 50) = 2 \times P\left(Z > \frac{50}{40} = 1.25\right) \simeq \underline{0.2112}$$

(ii) (a) REQ. $P\left(X(\tfrac{1}{2}) > -10\right) = P\left(Z > \frac{-10}{20} = -\frac{1}{2}\right) \simeq \underline{0.3085}$

(b) BROWNIAN BRIDGE $x_1 = 0 \quad x_2 = 60 \quad t = 1$

$t_1 = 0 \quad t_2 = 1\tfrac{1}{2}$

$$\text{MEAN} = \frac{0 + 60 \times 1}{1\frac{1}{2}} = 40$$

$$\text{VARIANCE} = \frac{\frac{1}{2} \times 1 \times 800}{1\frac{1}{2}} = \frac{800}{3}$$

REQUIRE $P\left(\frac{-30-40}{\sqrt{800/3}} < Z < \frac{30-40}{\sqrt{800/3}} \simeq -0.6124\right) \simeq \underline{0.2701}$

(iii) $$P(W_{60} > 2) = 1 - P(W_{60} < 2) = 1 - 2\left[1 - \phi\left(\frac{60}{\sqrt{1600}}\right)\right]$$

$$= 1 - 2\left[1 - \phi(1.5)\right] \simeq \underline{0.8664}$$

(iv)

Z	$20Z$	POSITION	TIME
–	–	0	2.00
1.2816	25.632	25.6	2.30
0.4435	8.87	34.5	3.00
0.3760	7.52	42.0	3.30
−0.6307	−12.614	29.4	4.00