M343 1998

CAVEAT EMPTOR (THERE WILL BE ERRORS!)

PART I

1. (i) MEAN = "$\frac{p}{q}$" $= \frac{1}{9}$ VARIANCE = "$\frac{p}{q^2}$" $= \underline{\underline{\frac{10}{81}}}$

(ii) $E(S(t)) = \eta \lambda_t \underset{t=6\text{ hrs}}{=} \frac{1}{9} \times 18 = \underline{\underline{2}}$ WHERE $S(t)$ IS THE TOTAL NO OF PURCHASES IN TIME t.

$$V(S(t)) = \lambda t(\eta^2 + \sigma^2) \underset{(t=6)}{=} 18\left(\frac{1}{81} + \frac{10}{81}\right) = \underline{\underline{\frac{22}{9}}}$$

(iii) $$I(t) = \frac{\sigma^2(t)}{\eta(t)} = \frac{\lambda_t(\eta^2+\sigma^2)}{\eta \lambda_t} = \eta + \frac{\sigma^2}{\eta} = \frac{1}{9} + \frac{10}{9} = \underline{\underline{\frac{11}{9}}}$$

$I(t) > 1$ SO PURCHASES ARE MORE VARIABLE THAN THEY WOULD BE IF THEY FOLLOWED A POISSON PROCESS

---

2. (i) $P(X_5 = -2) = \cancel{(\ }= 0$

$P(X_6 = 2) = \binom{6}{4} 0.7^4\, 0.3^2 \simeq \underline{\underline{0.324}}$

(ii) $f_8 = \frac{1}{7} u_8 = \frac{1}{7}\binom{8}{4} 0.7^4\, 0.3^4 \simeq \underline{\underline{0.0194}}$

(iii) GAMBLER'S RUIN

$p = 0.7$ $q = 0.3$ OPPONENT WITH UNLIMITED RESOURCES

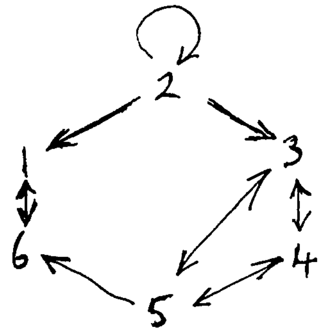
$j = 3$

$q_j = \left(\frac{q}{p}\right)^j = \left(\frac{3}{7}\right)^3 \simeq \underline{\underline{0.0787}}$

3 (i) $[0\ 0\ 1\ 0\ 0\ 0]P^2 = [0\ 0\ \tfrac{1}{2}\ \tfrac{1}{4}\ \tfrac{1}{8}\ \tfrac{1}{8}]$ so $\tfrac{1}{2}$

(ii) $p_{35}\,p_{54} = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$

(iii)



$\{2\}$ NOT CLOSED TRANSIENT APERIODIC

$\{1\ 6\}$ CLOSED RECURRENT PERIODIC PERIOD $= 2$

$\{3\ 4\ 5\}$ NOT CLOSED TRANSIENT APERIODIC

4 (a) (i) $M/U/4$ (ii) $D/\Gamma/1$

(b) (i) $\rho = \dfrac{\lambda}{n\mathcal{E}} = \dfrac{24}{1 \times 30} = 0.8$ Use

Use $M/M/1$

(ii) $K = 1 + \dfrac{0.8}{0.2} = 5$

$p_0 = \dfrac{1}{5} \quad p_1 = \dfrac{0.8}{5} \quad p_2 = \dfrac{(0.8)^2}{5} \ \ldots$

SO REQUIRED PROPORTION IS $p_1 = 0.16$

(iii) $E\left(\begin{smallmatrix}\text{NUMBER}\\ \text{IN QUEUE}\end{smallmatrix}\right) = \left(0 \times \tfrac{1}{5}\right) + \left(1 \times \tfrac{0.8}{5}\right) + \left(\tfrac{2 \times 0.8^2}{5}\right) + \ldots = 4$

(iv) QUEUING TIME $\sim M\left(\dfrac{1}{\mathcal{E} - \lambda}\right) = M\left(\cancel{\tfrac{1}{6}}\right)$ $M(0.1)$ (HOURS)

5 (i) $F(s) = 0.7s + 0.3s^2$

$U(s) = \dfrac{1}{1 - F(s)} = \left(1 - (0.7s + 0.3s^2)\right)^{-1}$

$= 1 + (0.7s + 0.3s^2) + (0.7s + 0.3s^2)^2 + (0.7s + 0.3s^2)^3 + \ldots$

$= 1 + 0.7s + (0.3 + 0.49)s^2 + (0.42 + 0.343)s^3 + \ldots$

so $u_2 = 0.79$

$u_3 = 0.763$

5. (CONT.)

(ii) W_6 HAS P.G.F $\left(F(s)\right)^6 = \left(0.7s + 0.3s^2\right)^6$

$$= s^6\left(0.7 + 0.3s\right)^6$$

SO $W_6 = 6 + K_6$ WHERE $K_6 \sim B(6, 0.3)$

$$P(W_6 = 10) = P(K_6 = 4) = \binom{6}{4}(0.3)^4(0.7)^2 \simeq \underline{\underline{0.0595}}$$

(iii) $$\frac{1}{\eta_T} = \frac{1}{F'(1)} = \frac{1}{0.7+0.6} = \underline{\underline{\frac{10}{13}}}$$

6. (i) REQ $P(250 + 10 + X(t) < 250)$

$$= P(X(t) < -10) = P\left(Z < \frac{-10}{\sqrt{400}} = \overset{-0.5}{\cancel{-1}}\right) \simeq \underline{\underline{\overset{0.385}{\cancel{0.1587}}}}$$

Because the Brownian bit is $X(t)$

(ii) BROWNIAN BRIDGE

$x_1 = \overset{0}{\cancel{250}}$ $t_1 = 0$ $x_2 = \overset{10}{\cancel{300}}$ $t_2 = 4$ $t = 1$

$$\text{MEAN} = \frac{\overset{0}{\cancel{250}}(4-\cancel{1}) + \overset{10}{\cancel{300}}(\cancel{1}-0)}{4} = \cancel{250+15} = \cancel{265}\ 2.5$$

$$\text{VARIANCE} = \frac{(t_2-t)(t-t_1)}{t_2-t_1}\sigma^2 = \frac{3\times 1}{4}400 = \cancel{300}$$

REQ $$P\left(Z < \frac{\overset{-10}{\cancel{250}} - \overset{0}{\cancel{265}}}{\sqrt{300}} = \overset{-0.722}{\cancel{-0.866}}\right) \simeq \underline{\underline{\cancel{0.193}}}\ \ 0.24$$

PART II

7 (a) e.g. DANDELIONS IN MY LAWN.
RANDOMLY-POSITIONED CLUSTERS
(REPRODUCTIVE CLUMPING)

(b) (i) $R \sim$ RAYLEIGH $\left(f(r) = 2\pi\times 120\, r\, e^{-\pi\times 120\times r^2}\right)$

$$E(R) = \frac{1}{\sqrt{240\pi}}\sqrt{\frac{\pi}{2}} \simeq 0.046\,\text{km} = \underline{\underline{45.6\,\text{m}}}.$$

$$P(R<30) = \int_0^{0.03} 240\pi\, r\, e^{-120\pi r^2}\,dr = \left[-e^{-120\pi r^2}\right]_0^{0.03}$$

$$\underline{\underline{\simeq 0.2877}}$$

7. (CONT)

(b) (ii) $120 \times 0.2 \times 0.15 = 3.6$

$$P(N \geq 2) = 1 - P(N=0) - P(N=1)$$
$$= 1 - e^{-3.6} - 3.6e^{-3.6} \simeq \underline{\underline{0.874}}$$

(iii) CUMULATIVE POISSON PROBS —

0	$e^{-3.6} \simeq 0.0273$
1	0.1257
2	0.3027
3	0.5152

SIMULATED NUMBER = 3

(iv) 2 UNIFORM 0/1 RANDOM VARIABLES FOR EACH SHRUB, SCALED BY 200 & 150 RESPECTIVELY TO GIVE COEFFICIENTS IN m.

(v) UNIFORM R.V. FOR EACH SIMULATED SHRUB IF < 0.3 ~ FLOWERING.

---

8. (a) (i) POPULATION WILL NOT BECOME EXTINCT

(ii) ULTIMATE EXTINCTION IS CERTAIN

(iii) " " " "

(iv) ULTIMATE EXTINCTION IS POSSIBLE, BUT NOT CERTAIN

(b) (i) $\mu_1 = 2 \qquad \sigma_1^2 = 3 \times \frac{2}{3} \times \frac{1}{3} = \underline{\underline{\frac{2}{3}}}$

$\mu_2 = 4 \qquad \sigma_2^2 = \dfrac{2 \cdot \frac{2}{3} \cdot (4-1)}{2-1} = \underline{\underline{4}}$

(ii) $\Pi(s) = \left(\frac{2}{3}s + \frac{1}{3}\right)^3$

at 3rd gen $\qquad \theta_3 = \Pi^3(0) \simeq \underline{\underline{0.04820}} \qquad \theta_2 \ \Pi^2(0)$

diff $\theta_3 - \theta_2 = \cdot 0023$

(iii) NEED TO SOLVE $\theta = \Pi(\theta)$

i.e. $\theta = \left(\frac{2}{3}\theta + \frac{1}{3}\right)^3$

EXPANDING & FACTORING OUT $(\theta - 1)$ GIVES $8\theta^2 + 20\theta - 1 = 0$

SMALLEST +VE SOL^N IS $\dfrac{-20 + \sqrt{432}}{16} \simeq \underline{\underline{0.0490}}$

8 (b) (CONT)

(iv) $\cancel{\pi^*(s) = \left[\left(\frac{2}{3}s + \frac{1}{3}\right)^3\right]^3}$

$$\left[\frac{-20 + \sqrt{432}}{16}\right]^3 = 0.00018$$

SOLVING $\pi^*(g) = g$ BY ITERATION GIVES $g = \cancel{0.00005085}$

9 (i)

$$v(1-s)\frac{\partial \pi}{\partial s} - \frac{\partial \pi}{\partial t} = 0$$

$$\frac{ds}{v(1-s)} = \frac{dt}{-1} = \frac{d\pi}{0}$$

$$c_1 = \pi$$

$$-\int \frac{ds}{1-s} = \int v\,dt + \text{CONST} \Rightarrow \ln(1-s) = vt + \text{CONST}$$

$$c_2 = (1-s)e^{-vt}$$

GEN SOLN. $\pi = \psi\left((1-s)e^{-vt}\right)$

INITIAL CONDITION — $\pi(s,0) = s^6$

SO $\psi(1-s) = s^6$ SO $\psi(x) = (1-x)^6$

SO $\pi(s,t) = \left(1 - (1-s)e^{-vt}\right)^6 = \left(e^{-vt}s + (1-e^{-vt})\right)^6$

SO $X(t) \sim \text{BIN}(6, e^{-vt})$

(ii) DIST. OF TIME TO FIRST DEATH IS $M(6v)$
SO EXPECTED TIME IS $\frac{1}{6v}$

EXPECTED TIME TO 6^{th} DEATH $= \frac{1}{6v} + \frac{1}{5v} + \frac{1}{4v} + \frac{1}{3v} + \frac{1}{2v} + \frac{1}{v}$

$$\simeq \frac{2.45}{v}$$

(iii) $1 - \pi(0,t) = 1 - (1-e^{-vt})^6$

10 (a) SIMILARITIES IN BOTH MODELS $\rho = \frac{n\gamma}{\beta}$ IS A THRESHOLD VALUE

WHEN $x_0 < \rho$ BEHAVIOUR SIMILAR:—

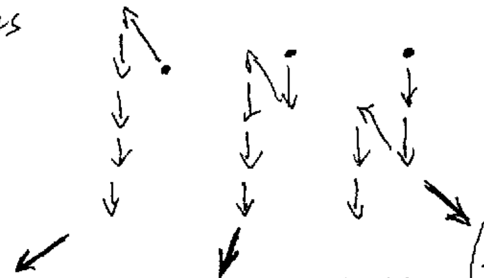
- DETERMINISTIC — EPIDEMIC QUICKLY DIES OUT
- STOCHASTIC — MINOR OUTBREAK WITH HIGH PROB.

DIFFERENCES WHEN $x_0 > \rho$

- DETERMINISTIC — GENUINE OUTBREAK
  NO. OF SURVIVORS $< \rho$
- STOCHASTIC — MINOR OR MAJOR OUTBREAK WITH HIGH PROB. INTERMEDIATE SIZE OUTBREAK UNLIKELY.

c ~~(b)~~ (i) $\rho = \frac{n\gamma}{\beta} = \frac{30 \cdot \frac{1}{4}}{1.5} = 5$

(ii) 3 WAYS



$$\frac{28}{33} \times \left(\frac{5}{32}\right)^4 \qquad \frac{5}{33} \times \frac{28}{33} \times \left(\frac{5}{32}\right)^3 \qquad \left(\frac{5}{33}\right)^2 \times \left(\frac{28}{33}\right) \times \left(\frac{5}{32}\right)^2$$

$$\text{PROB} = \frac{5^4 \times 28}{33 \times 32^2}\left(\frac{1}{32^2} + \frac{1}{33 \times 32} + \frac{1}{33^2}\right) \simeq \underline{\underline{0.00147}}$$

(iii) $P(\text{MAJOR OUTBREAK}) = 1 - \left(\frac{5}{28}\right)^3 \simeq \underline{\underline{0.9943}}$

11 (a) (i) POPULATION OF PET CATS IN WORTHING
~ AGE DISTRIBUTION CONSTANT
~ INTRINSIC GROWTH RATE ZERO

(ii) POPULATION OF TOMATO PLANTS IN MY GREENHOUSE
— AGE DISTRIBUTION NOT CONSTANT OVER TIME.

10(b)i Stoc. Simp Epidemic

ii $E(W) = \frac{1}{2\beta} + \frac{1}{2\beta} + \frac{3}{5\beta} + \frac{1}{\beta} = \frac{2.6}{\beta} = 2.6$ days ?

$Var \approx \left(\frac{1}{2\beta}\right)^2$ etc. $\qquad \frac{1.86}{\beta^2} \sim 2.30 (\text{days})^2$.

11 (b) (i) $$Q(x) = \exp\left(-\int_0^x \frac{3}{2+u}\,du\right)$$

$$= \exp\left(-\left[3\ln(2+u)\right]_0^x\right)$$

$$= \exp\left(-\ln\left[\left(1+\tfrac{x}{2}\right)^3\right]\right)$$

$$= \frac{1}{\left(1+\frac{x}{2}\right)^3} = \underline{\underline{\frac{8}{(2+x)^3}}}$$

(ii) $$e_0 = \int_0^\infty \frac{8}{(2+x)^3}\,dx = \left[\frac{-4}{(2+x)^2}\right]_0^\infty = \underline{\underline{1}}$$

(iii) $$g(x) = \frac{Q(x)}{e_0} = \frac{8}{(2+x)^3}$$

$$G(x) = \int_0^x g(u)\,du = \left[\frac{-4}{(2+u)^2}\right]_0^x = \underline{\underline{1 - \frac{4}{(2+x)^2}}}$$

(iv) (a) $$1 - G(3) = \frac{4}{25} = \underline{\underline{0.16}}$$

(b) $$Q(3) = \frac{8}{125} = \underline{\underline{0.064}}$$

(v) $$\mu = \int_0^\infty x\,g(x)\,dx = \int_0^\infty \frac{8x}{(2+x)^3}\,dx$$

$$\overset{\text{PARTS}}{=} \left[\frac{-4x}{(2+x)^2}\right]_0^\infty + \int_0^\infty \frac{4}{(2+x)^2}\,dx$$

$$= \left[-\frac{4}{(2+x)}\right]_0^\infty = \underline{\underline{2}}$$

12 (i) $q^2 = 0.36$ $\quad q = 0.6$ so $p = 0.4$

YELLOW — 16% GREEN — 48%

(ii)

GENOTYPE (GEN. 0)	AA	Aa	aa
	u	$2v$	w
	0	$48/84$	$36/84$
(SUBSEQUENT GENERATIONS)	p^2	$2pq$	q^2
	$\frac{4}{49}$	$\frac{20}{49}$	$\frac{25}{49}$

$$p = u + v = \frac{24}{84} = \frac{2}{7}$$

$$q = v + w = \frac{60}{84} = \frac{5}{7}$$

HARDY – WEINBERG LAW

(iii)

PARENTS		PROB	OFFSPRING + PROBS AA	Aa	aa
AA	AA	$16/49^2$	1	0	0
Aa	Aa	$400/49^2$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
aa	aa	$625/49^2$	0	0	1
AA	Aa	$160/49^2$	$\frac{1}{2}$	$\frac{1}{2}$	0
AA	aa	$200/49^2$	0	1	0
Aa	aa	$1000/49^2$	0	$\frac{1}{2}$	$\frac{1}{2}$

$$P\left(\text{PARENTS } Aa\ Aa \mid \text{OFFSPRING } Aa\right) = \frac{400 \times \frac{1}{2}}{\left(400 \times \frac{1}{2}\right) + \left(160 \times \frac{1}{2}\right) + 200 + \left(1000 \times \frac{1}{2}\right)}$$

$$= \frac{200}{980} = \frac{10}{49} \simeq 0.2041$$