Question 1 (i) (a)

Let X be the distance to the nearest flower. From the *Handbook* this has the Rayleigh distribution with p.d.f.

$$f(x) = 2\pi\lambda x \exp(-\pi\lambda x^2)$$

and parameter $\beta = \sqrt{2\pi\lambda} = \sqrt{2\pi \times 0.15} = \sqrt{0.3\pi} =$. Therefore

$$E(X) = \frac{1}{\beta}\sqrt{\frac{\pi}{2}} = \frac{1}{\sqrt{0.3\pi}}\sqrt{\frac{\pi}{2}} = \frac{1}{\sqrt{0.6}} \simeq 1.2910$$

(b)

$$\begin{aligned}
P(X > 2) &= 1 - P(X \leq 2) \\
&= 1 - F(2) \\
&= 1 - (1 - exp(-\beta^2 x^2/2)) \\
&= \exp(-\beta^2 x^2/2) \\
&= \exp(-2\beta^2) \\
&= \exp(-2 \times 0.3\pi) \\
&= \exp(-0.6\pi) \\
&\simeq 0.1518
\end{aligned}$$

(ii) (a)

Let the mean number of flowers in this region be Y. Now

$$Y \sim \text{Poisson}(\text{Area} \times \lambda) = \text{Poisson}(12 \times 0.15) = \text{Poisson}(1.8)$$

Therefore $P(Y = 0) = \exp(-1.8) \simeq 0.1653$

(b)

$$
\begin{aligned}
P(X > 2) &= 1 - P(X \leq 2) \\
&= 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \\
&= 1 - \left[\exp(-1.8) + \frac{\exp(-1.8) \times 1.8}{1!} + \frac{\exp(-1.8) \times 1.8^2}{2!}\right] \\
&= 1 - \exp(-1.8)\left[1 + 1.8 + \frac{\times 1.8^2}{2!}\right] \\
&= 1 - 4.42\exp(-1.8) \\
&\simeq 1 - 0.7306 \\
&= 0.2694
\end{aligned}
$$

(iii)

$$
P(X = 3) = \frac{\exp(-1.8) \times 1.8^3}{3!} \simeq 0.1607
$$

$$
P(X \leq 3) = P(X \leq 2) + P(X = 3) \simeq 0.7306 + 0.1607 = 0.8913
$$

$$
P(X \leq 2) = 0.7306 < 0.8872 < 0.8913 < P(X \leq 2 + 1) < P(X \leq 3)
$$

The simulated value is $X = 3$

Question 2 (i)

This is the modified geometric distribution with parameters

$$a = 2, b = -3, c = 8, d = 3$$

and from the *Handbook* the number of individuals in the first generation is

$$E(X) = \frac{ad - bc}{(c-d)^2} = \frac{2 \times 3 + 3 \times 8}{(8-3)^2} = \frac{6 + 24}{5^2} = \frac{30}{25} = \frac{6}{5}$$

(ii)

Let the probability that the population becomes extinct by the nth generation be θ_n and

$$\begin{aligned} \theta_n &= \Pi(\theta_{n-1}) \\ \theta_1 = \Pi(0) &= \frac{2 + 3 \times 0}{8 - 3 \times 0} = 0.25 \\ \theta_2 = \Pi(\theta_0) &= \Pi(0.25) \simeq 0.37931 \\ \theta_3 = \Pi(\theta_1) &= \Pi(0.37931) \simeq 0.45729 \end{aligned}$$

The probability that the population becomes extinct at the third generation is

$$\theta_3 - \theta_2 \simeq 0.45729 - 0.37931 \simeq 0.07798$$

(iii)

The probability that the population eventually becomes extinct is θ^* and is given by the smallest positive solution of

$$\begin{aligned} & & \theta &= \Pi(\theta) \\ & & &= \frac{2 + 3\theta}{8 - 3\theta} \\ \Rightarrow & & \theta(8 - 3\theta) &= 2 + 3\theta \\ \Rightarrow & & 8\theta - 3\theta^2 &= 2 + 3\theta \\ \Rightarrow & & 0 &= 3\theta^2 - 5\theta + 2 \\ \Rightarrow & & 0 &= (3\theta - 2)(\theta - 1) \\ \Rightarrow & & \theta &= \frac{2}{3} \text{ or } \theta = 1 \end{aligned}$$

Therefore $\theta^* = \frac{2}{3}$. I knew that this would be less than one since the mean number of offspring from any individual in one generation is greater than 1.

Question 3

$$
\begin{aligned}
&& \frac{\partial \Pi}{\partial t} &= -\beta s(1-s)\frac{\partial \Pi}{\partial s} \\
\Rightarrow && \beta s(1-s)\frac{\partial \Pi}{\partial s} + \frac{\partial \Pi}{\partial t} &= 0
\end{aligned}
$$

The auxiliary equations are

$$
\frac{ds}{\beta s(1-s)} = \frac{dt}{1} = \frac{d\Pi}{0}
$$

Taking the last two

$$
\begin{aligned}
&& 0 \cdot dt &= 1 \cdot d\Pi \\
\Rightarrow && \Pi &= c_1
\end{aligned}
$$

Taking the first two

$$
\begin{aligned}
&& \frac{ds}{\beta s(1-s)} &= \frac{dt}{1} \\
\Rightarrow && \frac{ds}{s(1-s)} &= \beta dt \\
\Rightarrow && \int \frac{ds}{s(1-s)} &= \int \beta\, dt \\
\Rightarrow && \int \left(\frac{1}{s} + \frac{1}{1-s}\right) ds &= \int \beta\, dt \\
\Rightarrow && \log s - \log(1-s) &= c_2 + \beta t \\
\Rightarrow && \log \frac{s}{1-s} &= c_2 + \beta t \\
\Rightarrow && \frac{s}{1-s} &= c_2 e^{\beta t} \\
\Rightarrow && c_2 &= e^{-\beta t}\frac{s}{1-s} \\
&& c_1 &= \psi(c_2) \\
&& \Pi(s,t) &= \psi\left(e^{-\beta t}\frac{s}{1-s}\right)
\end{aligned}
$$

(ii)

At $t = 0$ there are two individuals alive and so $\Pi(s, 0) = s^2$.

$$\Pi(s,0) = \psi\left(e^{-\beta\cdot 0}\frac{s}{1-s}\right) = \psi\left(\frac{s}{1-s}\right) = s^2$$

Now let

$$x = \frac{s}{1-s} \Leftrightarrow x(1-s) = s \Leftrightarrow x = s + sx \Leftrightarrow x = s(1+x) \Leftrightarrow s = \frac{x}{1+x}$$

so

$$\psi(x) = s^2 = \left(\frac{x}{1+x}\right)^2$$

$$\begin{aligned}\Pi(s,t) &= \psi\left(\frac{s}{1-s}e^{-\beta t}\right)\\ &= s^2\\ &= \left(\frac{\frac{s}{1-s}e^{-\beta t}}{1+\frac{s}{1-s}e^{-\beta t}}\right)^2\\ &= \left(\frac{e^{-\beta t}s}{1-(1-e^{-\beta t})s}\right)^2\end{aligned}$$

Question 4 (i)

$$\rho = \frac{n\gamma}{\beta} = \frac{(6-1\,1}{\times}\frac{1}{4}1.25 = \frac{5}{4} \cdot \frac{4}{5} = 1$$

(ii)

There are four paths which result in no more than two new infections. These may be represented as

$$\begin{aligned}
A &= (5,1) \to (5,0) \\
B &= (5,1) \to (4,2) \to (4,1) \to (4,0) \\
C &= (5,1) \to (4,2) \to (3,3) \to (3,2) \to (3,1) \to (3,0) \\
D &= (5,1) \to (4,2) \to (4,1) \to (3,2) \to (3,1) \to (3,0)
\end{aligned}$$

$$\begin{aligned}
P(A) &= \frac{1}{1+5} = \frac{1}{6} \\
P(B) &= \frac{5}{1+5} \times \frac{1}{1+4} \times \frac{1}{1+4} = \frac{5}{6 \cdot 5 \cdot} = \frac{1}{30} \\
P(C) &= \frac{5}{1+5} \times \frac{4}{1+4} \times \frac{1}{1+3} \times \frac{1}{1+3} \times \frac{1}{1+3} = \frac{5 \cdot 4}{6 \cdot 5 \cdot 4 \cdot 4 \cdot 4} = \frac{1}{96} \\
P(D) &= \frac{5}{1+5} \times \frac{1}{1+4} \times \frac{4}{1+4} \times \frac{1}{1+3} \times \frac{1}{1+3} = \frac{5 \cdot 4}{6 \cdot 5 \cdot 5 \cdot 4 \cdot 4} = \frac{1}{120}
\end{aligned}$$

The probability that there will be no more than two new infections is

$$\frac{1}{6} + \frac{1}{30} + \frac{1}{96} + \frac{1}{120} = 0.21875$$

Question 5 (i)

6.27 days

(ii)

$$e^{-x^2/50}$$

(iii)

5.89 days

(iv)

3.98 days

(v)

0.1352

Question 6 (i)

$$P(V(0.5) > 1.25V(0)) \simeq 1 - phi(0.8141) \simeq 0.1860$$

(ii)

$$\begin{aligned}
& & V\left(\frac{1}{3}\right) &< \frac{3}{4}V(0) \\
&\Leftrightarrow & \exp\left(X\left(\frac{1}{3}\right)\right) &< \frac{3}{4}\exp\left(X(0)\right) \\
&\Leftrightarrow & \exp\left(X\left(\frac{1}{3}\right)\right) &< \exp\left(\log\left(\frac{3}{4}\right)\right)\exp\left(X(0)\right) \\
&\Leftrightarrow & \exp\left(X\left(\frac{1}{3}\right)\right) &< \exp\left(X(0) + \log\left(\frac{3}{4}\right)\right) \\
&\Leftrightarrow & X\left(\frac{1}{3}\right) &< X(0) + \log\left(\frac{3}{4}\right) \\
&\Leftrightarrow & X\left(\frac{1}{3}\right) &< X(0) + \log\left(\frac{3}{4}\right) \\
&\Leftrightarrow & X\left(\frac{1}{3}\right) - X(0) &< \log\left(\frac{3}{4}\right)
\end{aligned}$$

$$X\left(\frac{1}{3}\right) \sim N\left(X(0), \frac{(1-1/3)(1/3-0)}{1-0}\sigma^2\right) = N\left(X(0), \frac{2}{9}\cdot\frac{1}{8}\right) = N\left(X(0), \frac{1}{36}\right)$$

$$\begin{aligned}
P\left(V\left(\frac{1}{3}\right) < \frac{3}{4}V(0)\right) &= P\left(X\left(\frac{1}{3}\right) - X(0) < \log\left(\frac{3}{4}\right)\right) \\
&= \phi\left(6\log\left(\frac{3}{4}\right)\right) \\
&\simeq \phi(-1.7261) \\
&\simeq 0.0422
\end{aligned}$$

Question 7 (a) (i)

Cars on a quiet motorway, passing a bridge at random intervals hence the number of cars in each minute has a Poisson distribution. Each car carries one or more passengers, which is a variate with an unknown discrete distribution. The number of passengers passing the bridge in each minute has a compound Poisson distribution.

(ii)

Vehicles passing a bridge on a quiet motorway are classified as car, HGV or motorbike. It is reasonable to assume that the vehicles pass the bridge at random intervals and that the type of any one vehicle is independent of the preceding vehicle.

(b)

Let $X(t)$ be the number of errors in the first t hours. $X(t) \sim \text{Poisson}(\mu(t))$

$$\begin{aligned}
\mu(t) &= \int_0^t \frac{3}{2+3u} du \\
\mu(t) &= \int_0^t \frac{1}{v} \cdot \frac{dv}{du} du \\
&= [\log v]_0^t \\
&= [\log(2+3u)]_0^t \\
&= [\log(2+3t) - \log 2] \\
&= \log \frac{2+3t}{2}
\end{aligned}$$

$$E(X) = \mu(t) = \log\left(\frac{2}{3} + t\right)$$

(ii)

Let Y be the number of errors made in the first hour.

$$\begin{aligned} Y &\sim \text{Poisson}(\mu(1)) \\ &= \text{Poisson}\left(\log\left(\frac{2}{3}+1\right)\right) \\ &= \text{Poisson}\left(\log\left(\frac{5}{3}\right)\right) \\ P(Y=0) &= \exp(-\log(5/3) \\ &= 3/5 \end{aligned}$$

(iii)

Let Z be the number of errors made in the second hour.

$$\begin{aligned} Z &\sim \text{Poisson}(\mu(2)-\mu(1)) \\ &= \text{Poisson}\left(\log\left(\frac{2}{3}+2\right)-\log\left(\frac{2}{3}+1\right)\right) \\ &= \text{Poisson}\left(\log\left(\frac{8}{5}\right)\right) \\ P(Z>0) &= 1-P(Z=0) \\ &= 1-\exp(-\log(8/5)) \\ &= 1-5/8 \\ &= 3/8 \end{aligned}$$

(iv)

If the programmer is to make her first serious error in the second hour then she must not make any errors in the first hour but at least one in her second hour. These are independent events and so

$$P(\text{first error in second hour}) = 3/5 \times 3/8 = 9/40$$

(v)

The jth event occurs at time t_j. The time to the next event has c.d.f.

$$\begin{aligned} F(t) &= 1-\exp(-\mu(t_j,t)) \\ &= 1-\exp\left(-\log\left(1+\frac{3}{2}t\right)-\log\left(1+\frac{3}{2}t_j\right)\right) \\ &= 1-\exp\left(\log\left(\frac{2+3t_j}{2+3t}\right)\right) \\ &= 1-\frac{2+3t_j}{2+3t} \end{aligned}$$

I will simulate the time of the $(j+1)$th event by finding t such that $F(t) = u$.

$$\begin{aligned}
u &= 1 - \frac{2+3t_j}{2+3t_{j+1}} \\
\frac{2+3t_j}{2+3t_{j+1}} &= 1 - u \\
\frac{2+3t_j}{u-1} &= 2+3t_{j+1} \\
3t_{j+1} &= \frac{2+3t_j}{1-u} - 2 \\
3t_{j+1} &= \frac{2+3t_j-2(1-u)}{1-u} \\
3t_{j+1} &= \frac{3t_j+2u}{1-u} \\
t_{j+1} &= \frac{3t_j+2u}{3(1-u)}
\end{aligned}$$

(vi)

$$\begin{aligned}
j_1 = \frac{3t_0+2u_1}{3(1-u_1)} &= \frac{3 \times 0 + 2 \times 0.23735}{3(1-0.23735} = \frac{2 \times 0.23735}{3(1-0.23735} \simeq 0.20748 \simeq 12 \text{ minutes} \\
j_2 = \frac{3t_1+2u_2}{3(1-u_2)} &= \frac{3 \times 0.20748 + 2 \times 0.49200}{3(1-0.49200} \simeq 1.05409 \simeq 63 \text{ minutes}
\end{aligned}$$

Question 8

(a)

Insert example from Units here ☺

(b) (i)

Let Y_n be the number of rightward steps of the total n steps. Now

$$Y_n \sim \text{Binomial}(n,p) \Rightarrow P(Y_n = y) = \binom{n}{y} p^y q^{n-y}$$

$X_n = k = \text{steps right} - \text{steps left} = y_n - (n - y_n) = 2y_n - n \Rightarrow y_n = (n+k)/2$

$$X_n \sim \text{Binomial}(n,p) \Rightarrow P(Y_n = y) = \binom{n}{y} p^y q^{n-y}$$

$$\begin{aligned}
P(X_n = k) &= P(Y_n = y) \\
&= \binom{n}{y} p^y q^{n-y} \\
&= \binom{n}{(n+k)/2} p^{(n+k)/2} q^{n-(n+k)/2} \\
&= \binom{n}{(n+k)/2} p^{(n+k)/2} q^{(n-k)/2}, \quad k \in \{-n, 2-n, \ldots, n-2, n\}
\end{aligned}$$

(ii)

After 7 steps the possible positions of the particle are $\{-7, -5, -3, -1, 1, 3, 5, 7\}$. Therefore $P(X_7 = 2) = 0$.

$$\begin{aligned}
P(X_7 = 1) &= \binom{7}{(7+1)/2} p^{(7+1)/2} q^{(7-1)/2} \\
&= \binom{7}{4} p^4 q^3 \\
&= \frac{7}{3!4!} p^4 q^3 \\
&= 35 p^4 q^3
\end{aligned}$$

(iii)

[Sorry, can't work out the 'textbook' way of doing this.]

If the particle is to return to its starting position for the first time after six steps there are only four possible motions in between these two times. At its first step it moves away from the starting point, then must move away again. By symmetry its last two steps are in the same direction. The possibilities for the paths are RR??LL and LL??RR. There are two possibilities for the middle two steps in each case giving four possible paths overall. The probability of each path is p^3q^3. Each path is mutually exclusive so the probability that the first return is after six steps is $4p^3q3$

Question 9

(b)

We require the probability $p_{DD}^{(2)}$.

$$\begin{aligned} p_{DD}^{(2)} &= p_{DH} \times p_{HD} + p_{DA} \times p_{AD} + p_{DD} \times p_{DD} \\ &= \frac{1}{2} \times 0 + \frac{1}{2} \times \frac{1}{4} + 0 \times 0 \\ &= \frac{1}{8} \end{aligned}$$

The probability that he is depressed on Monday, apathetic on Tuesday and happy on Wednesday is

$$p_{DA} \times p_{AH} = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

(ii)

Firstly note that the Markov chain is irreducible. I need to find the limiting distribution

$$\pi = \pi \mathbf{P}$$

$$\begin{aligned} \left[\begin{array}{ccc} \pi_H & \pi_A & \pi_D \end{array} \right] &= \left[\begin{array}{ccc} \pi_H & \pi_A & \pi_D \end{array} \right] \left[\begin{array}{ccc} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{array} \right] \\ &= \left[\begin{array}{ccc} \frac{2}{15} & \frac{8}{15} & \frac{1}{3} \end{array} \right] \end{aligned}$$

(iii)

By the Basic Limit Theorem

$$\mu_H = \frac{1}{\pi_H} = \frac{15}{2}$$

(iv)

$$\begin{aligned} &\frac{7}{16} \\ &\frac{23}{64} \\ &\simeq \frac{1}{3} \end{aligned}$$

Question 10 (i)

$$M/M/2$$

(ii)

$$p_0 = \frac{1}{K} = \frac{1}{9}$$

(iii)

The proportion of customers seen immediately is

$$p_0 + p_1 = \frac{1}{9} + \frac{8}{45} = \frac{13}{45}$$

(iv)

If more than three people are waiting then the queue size is > 5.

$$\begin{aligned} P(X > 5) &= 1 - P(X \leq 5) \\ &= 1 - (p_0 + p_1 + p_2 + p_3 + p_4 + p - p_5) \\ &\simeq 1 - 0.70873 \\ &= 0.29127 \end{aligned}$$

(v)

Expected queue size $= \Pi'(1) = \frac{200}{45} \simeq 0.4444$

Question 11 (a) (i) (a)

$$0.9 \times 0.9 = 0.81$$

(b)

$$2 \times 0.9 \times 0.1 = 0.18$$

(c)

$$0.1 \times 0.1 = 0.01$$

(ii) (a)

$$\begin{aligned} P(Aa|AA \cup Aa) &= \frac{P(Aa \cap (AA \cup Aa))}{P(AA \cup Aa)} \\ &= \frac{P(Aa)}{P(AA \cup Aa)} \\ &= \frac{0.18}{0.18 + 0.81} \\ &= \frac{0.18}{0.99} \\ &\simeq 0.1818 \end{aligned}$$

(ii) (b)

$$\begin{aligned} P(AA|AA \cup Aa) &= \frac{P(AA \cap (AA \cup Aa))}{P(AA \cup Aa)} \\ &= \frac{P(AA)}{P(AA \cup Aa)} \\ &= \frac{0.81}{0.18 + 0.81} \\ &= \frac{0.81}{0.99} \\ &\simeq 0.8182 \end{aligned}$$

(iii)

Ralph=R, Susan=S, Tom=T, Ursula=U.

$$\begin{aligned}
P(T = aa) &= P(T = aa|R = Aa \cup S = Aa) \times P(R = Aa \cup S = Aa) \\
&= P(T = aa|R = Aa \cup S = Aa) \times P(R = Aa) \times P(S = Aa) \\
&= 0.25 \times 0.1818^2 \\
&= 0.25 \times 0.1818^2 \\
&= 0.00826
\end{aligned}$$

(iv)

$$\begin{aligned}
&P(T = aa \cap (U = AA \cup U = Aa)) \\
&= P((U = AA \cup U = Aa)|T = aa) \times P(T = aa) \\
&= P((U = AA \cup U = Aa)|(R = Aa \cap S = Aa) \times P(T = aa) \\
&\simeq (0.25 + 0.5 \times 0.00826) \\
&\simeq 0.0062
\end{aligned}$$

(b) (i)

$$\begin{aligned}
P(J = b) &= P(J = b|I = Bb) \times P(I = Bb|I = BB \cup I = Bb) \\
&= 0.5 \times 0.1818 \\
&= 0.0909
\end{aligned}$$

(ii)

$$\begin{aligned}
P(I = Bb|J = B) &= \frac{P(J = B|I = Bb) \times P(I = Bb)}{P(J = B)} \\
&= \frac{0.5 \times 0.18}{0.9} \\
&= \frac{0.5 \times 0.1818}{0.9} \\
&= \frac{0.5 \times 0.1818}{0.9} \\
&= 0.1010
\end{aligned}$$

Question 12 (a)

(b) (i)

$$\begin{aligned}
H(t) &= \int_0^t h(u)\,du \\
&= \int_0^t \frac{2u}{25-u^2}\,du \\
&= -\int_0^t \frac{1}{25-u^2}(-2u)\,du \\
&= -\int_0^{u=t} \frac{1}{v}\cdot\frac{dv}{du}\,du, \quad v = 25-u^2 \\
&= -\int_0^{u=t} \frac{dv}{v} \\
&= -\left[\log v\right]_0^{u=t} \\
&= -\left[\log(25-u^2)\right]_0^t \\
&= -\left[\log(25-t^2) - \log 25\right] \\
&= \log\frac{25}{25-t^2}
\end{aligned}$$

Now the survivor function is

$$\begin{aligned}
Q(t) &= \exp(-H(t)) \\
&= \frac{25-u^2}{25}, \quad 0 \le t \le 5
\end{aligned}$$

The proportion of thermocouples lasting less than 2 years is

$$\begin{aligned}
G(2) &= 1 - Q(2) \\
&= 1 - \frac{25-2^2}{25} \\
&= 1 - \frac{25-4}{25} \\
&= 1 - \frac{21}{25} \\
&= \frac{4}{25}
\end{aligned}$$

(ii)

$$\begin{aligned}
\text{Mean lifetime } &= \int_0^\infty (1 - G(t))\,dt \\
&= \int_0^5 Q(t)\,dt \\
&= \int_0^5 \frac{25 - t^2}{25}\,dt \\
&= \left[t - \frac{1}{75}t^3\right]_0^5 \\
&= 5 - \frac{125}{75} \\
&= 5 - \frac{5}{3} \\
&= \frac{10}{3} \text{ years.}
\end{aligned}$$

(iii)

The residual lifetime, v, of the component has p.d.f.

$$\begin{aligned}
f_V(v) &= \frac{1 - G(v)}{\mu_T} \\
&= \frac{Q(v)}{\mu_T} \\
&= \frac{25 - v^2}{25} \cdot \frac{3}{10} \\
&= \frac{25 - v^2}{25} \cdot \frac{3}{10} \\
F_V(v) &= \int_0^v f_V(t)\,dt \\
&= \frac{3}{10}\int_0^v \frac{25 - t^2}{25}\,dt \\
&= \frac{3}{10}\left[t - \frac{1}{75}t^3\right]_0^v \\
&= \frac{3}{10}\left(v - \frac{1}{75}v^3\right)
\end{aligned}$$

The probability that this component fails within a year is

$$\begin{aligned}
F_V(1) &= \frac{3}{10}\left(1-\frac{1}{75}1\right)\\
&= \frac{3}{10}\left(1-\frac{1}{75}\right)\\
&= \frac{3}{10}\frac{74}{75}\\
&= \frac{74}{250}
\end{aligned}$$

(iv)

The total lifetime, W, of the component has p.d.f.

$$\begin{aligned}
f_W(w) &= \frac{wg(w)}{\mu_T} \\
&= \frac{w}{\mu_T} \cdot \frac{d}{dw} G(w) \\
&= \frac{w}{\mu_T} \cdot \frac{d}{dw} (1 - Q(w)) \\
&= \frac{w}{\mu_T} \cdot \frac{d}{dw} \left(1 - \frac{25 - w^2}{25}\right) \\
&= \frac{w}{\mu_T} \cdot \frac{d}{dw} \left(\frac{w^2}{25}\right) \\
&= \frac{w}{\mu_T} \cdot \frac{2w}{25} \\
&= \frac{2w^2}{25\mu_T} \\
&= \frac{6w^2}{250} \\
E(W) &= \int_0^\infty wf(w)\,dw \\
&= \int_0^5 w\frac{6w^2}{250}\,dw \\
&= \int_0^5 \frac{6}{250}w^3\,dw \\
&= \left[\frac{6}{1000}w^4\right]_0^5 \\
&= \left[\frac{6}{1000}w^4\right]_0^5 \\
&= \frac{6 \cdot 625}{1000} \\
&= \frac{6 \cdot 25}{40} \\
&= \frac{150}{40} \\
&= \frac{15}{4} \text{ years.}
\end{aligned}$$

At the moment that sampling occurs it is more likely that a longer lived component will be in use.

(v)

The expected residual lifetime of the component in place, $E(V)$, is

$$\begin{aligned}
E(V) &= \int_0^\infty v f_V(v)\,dv \\
&= \int_0^5 v\left(\frac{25-v^2}{25}\cdot\frac{3}{10}\right)\,dv \\
&= \frac{3}{10}\int_0^5 \left(v-\frac{1}{25}v^3\right)\,dv \\
&= \frac{3}{10}\left[\frac{1}{2}v^2-\frac{1}{100}v^4\right]_0^5 \\
&= \frac{3}{10}\left(\frac{25}{2}-\frac{625}{100}\right) \\
&= \frac{3}{10}\left(\frac{25}{2}-\frac{25}{4}\right) \\
&= \frac{3}{10}\cdot\frac{25}{4} \\
&= \frac{75}{40} \\
&= \frac{15}{8}
\end{aligned}$$

The expected number of new components required is

$$\frac{6}{\mu_T}=\frac{6}{10/3}=1.8$$